

HEEGAARD FLOER HOMOLOGY OF SURGERIES ON TWO-BRIDGE LINKS

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ABSTRACT. We give an $O(p^2)$ time algorithm to compute the generalized Heegaard Floer complexes $A_{s_1, s_2}^-(\vec{L})$'s for a two-bridge link $\vec{L} = b(p, q)$ by using nice diagrams. Using the link surgery formula of Manolescu-Ozsváth, we also show that \mathbf{HF}^- and their d -invariants of all integer surgeries on two-bridge links are determined by $A_{s_1, s_2}^-(\vec{L})$'s. We obtain a polynomial time algorithm to compute \mathbf{HF}^- of all the surgeries on two-bridge links, with $\mathbb{Z}/2\mathbb{Z}$ coefficients. In addition, we calculate some examples explicitly: the \mathbf{HF}^- and their d -invariants of all integer surgeries on a family of links.

1. INTRODUCTION

1.1. Background and motivation. Heegaard Floer homology is a package of invariants of 3-manifolds invented by Ozsváth and Szabó, using holomorphic disks and Heegaard splittings of the 3-manifold [12, 11]. It detects the Thurston norm and fiberedness of a 3-manifold [3, 10, 9]. Furthermore, it fits into a kind of 3+1 dimensional topological quantum field theory, which is important in the study of smooth structures on 4-manifolds. Unlike other Floer homological invariants, Heegaard Floer homology is combinatorially computable, and there are several algorithms for computing various versions of it. Manolescu, Ozsváth and Sarkar described knot Floer homology combinatorially using grid diagrams in [6]. Sarkar and Wang in [17] found an algorithm for computing $\widehat{HF}(M^3)$ over $\mathbb{Z}/2\mathbb{Z}$ by using nice Heegaard diagrams. Lipshitz, Ozsváth and Thurston used bordered Floer homology to give another algorithm for computing $\widehat{HF}(M^3)$ in [4]. In [7], Manolescu, Ozsváth and Thurston showed that the plus and minus versions of Heegaard Floer homology (over $\mathbb{Z}/2\mathbb{Z}[[U]]$) can also be described combinatorially, by using link surgery and grid diagrams. (Admittedly, the MOT algorithm has a high time complexity.) Improving these algorithms and developing new methods for computations are still important and interesting questions.

1.2. The basic idea. This paper is aimed at studying the surgeries on two-component links by using the link surgery formula for Heegaard Floer homology due to Manolescu-Ozsváth [5]. When \vec{L} is a two-bridge link $b(p, q)$, we find a fast algorithm for computing the Floer homology of surgeries on L , $\mathbf{HF}^-(S_\Lambda^3(\vec{L}))$ over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, where Λ is the framing matrix of a surgery. (Here, $\mathbf{HF}^-(S_\Lambda^3(\vec{L}))$ is the U -completion of HF^- . See [5] Section 2.) This algorithm uses genus-0 nice diagrams and algebraic arguments to simplify the Manolescu-Ozsváth surgery formula. Its time complexity is a polynomial of p and $\det(\Lambda)$.

Let us mention some related work. In [16], Rasmussen studied Heegaard Floer homology of surgeries on two-bridge knots. In [15], Ozsváth and Szabó developed a formula for the Heegaard Floer homology of surgeries on knots. The paper [5] presents a generalization of this formula to the case of links. Two sets of data are needed in the surgery formula in [5]: the generalized Floer complexes $A_s^-(\vec{L})$'s and the maps in the surgery formula, namely the maps $I_s^{\vec{L}'}, D_s^{\vec{L}'}$ connecting the complexes associated to oriented sublinks. In general, the Heegaard Floer homology of link surgeries is more difficult to compute, due to more involved algebraic structures. However, in some cases, computations using this surgery formula can be simplified.

The main complexity in the link surgery formula is the counting of the holomorphic domains on the Heegaard surface, which corresponds to holomorphic bigons and polygons in the symmetric product. For the special case of two-bridge links, we directly find a formula for the counts of

holomorphic bigons. Furthermore, the general link surgery formula involves counting holomorphic polygons in the symmetric product for computing some cobordism maps, and this is of considerably high time complexity. Here we notice that, for two-bridge links, all these maps can be determined algebraically.

1.3. Main results and organization. In section 2, we review some preliminaries for the link surgery formula, including the generalized Floer complexes, polygon maps and nice diagrams.

In section 3, using the Schubert normal form of two-bridge links we get a class of nice Heegaard diagrams called Schubert Heegaard diagrams, in which every region is either a bigon or a square. We can explicitly describe all the composite bigons on a Schubert Heegaard diagram, and hence the Floer differential. Further, we get a formula for the Alexander gradings of all intersection points, thus giving a formula for the multi-variable Alexander polynomial of a two-bridge link $b(p, q)$ in terms of p, q . See Theorem 3.18 and Proposition 3.13 below for the precise statements. This implies that $A_s^-(\vec{L})$ can be directly computed from this diagram. For a two-bridge link $\vec{L} = b(p, q)$, we get an $O(p^2)$ time algorithm for computing $A_s^-(\vec{L})$. We also found different two-bridge links (modulo mirror and reorientation) sharing the same multi-variable Alexander polynomial, signature, and linking number.

In section 4, we review the link surgery formula from [5] for two-component links $\vec{L} = \vec{L}_1 \cup \vec{L}_2$ with basic diagrams. First, we review some algebraic tools, *hyperboxes of chain complexes*. In order to see the algebraic structure of the link surgery formula, we define a *twisted gluing* of squares of chain complexes. Then the link surgery formula is a twisted gluing of certain squares of chain complexes derived from L . These squares are constructed in [5] by means of *complete system of hyperboxes*, which is a set of compatible Heegaard diagrams for the sublinks. For any two-component link, we define a type of complete system of hyperboxes which generalizes the basic systems used in [5], called a *primitive system of hyperboxes*. We also show that any basic diagram of $\vec{L} = \vec{L}_1 \cup \vec{L}_2$ produces a primitive system.

In section 5, we use algebraic arguments to show some rigidity results of the destabilization maps $D_s^{\vec{M}}$'s, $M \subset L$ up to chain homotopy, for two-bridge links. Further, if we perturb the destabilization maps $D_s^{\pm L_i}$'s by chain homotopy, i.e. replace $D_s^{\pm L_i}$ by $\tilde{D}_s^{\pm L_i} \simeq D_s^{\pm L_i}$, we can construct a new square of chain complexes called the *perturbed surgery complex*. Using the rigidity results, we show that the perturbed surgery complex is isomorphic to the original complex in the link surgery formula. Based on the perturbed surgery complex, we give the algorithm for computing $\mathbf{HF}^-(S_\Lambda^3(\vec{L}))$ mentioned before.

Throughout this paper, we use $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients. The main result we obtain is the following:

Theorem 1.1. *Suppose \vec{L} is an oriented two-bridge link with framing Λ . Let \mathcal{H}^L be a basic Heegaard diagram of \vec{L} and let \mathcal{H} be a primitive system induced by \mathcal{H}^L . After we determine the modules $A_s^-(\vec{L})$'s sitting at the vertices of the hypercube in the link surgery formula, any choices of*

- *homotopy equivalences $\tilde{D}_{s_1, s_2}^{-L_i}$ as $\mathbb{F}[[U_1, U_2]]$ -modules for the edge maps, and*
- *$\mathbb{F}[[U_1]]$ -linear chain homotopies for the diagonal maps in the hypercube*

yield a perturbed surgery complex $(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$ which is isomorphic to the original surgery complex in [5] as an $\mathbb{F}[[U_1]]$ -module. By imposing the U_2 -action to be the same as the U_1 -action, the $\mathbb{F}[[U_1, U_2]]$ -module $H_(\tilde{\mathcal{C}}^-(\mathcal{H}, \Lambda), \tilde{\mathcal{D}}^-)$ becomes isomorphic to the homology $\mathbf{HF}^-(S_\Lambda^3(\vec{L}))$. This isomorphism preserves the grading.*

Corollary 1.2. *For a two-bridge link \vec{L} , knowledge of the $A_s^-(\vec{L})$ determines \mathbf{HF}^- of all the surgeries on L .*

In section 6, we compute some examples explicitly: the surgeries on $b(4n, 2n+1)$, $n \in \mathbb{N}$. First, we prove that when $L = b(4n, 2n+1)$, the filtered chain homotopy type of $CFL^-(L)$ is determined

by the filtered chain homotopy type of $\widehat{CFL}(L)$. This is based on an observation that the Alexander polytope is simple and there are several symmetries on $CFL^-(L)$, which give constraints for the differentials in $CFL^-(L)$. From $CFL^-(L)$ we derive all the $A_s^-(L)$'s and the inclusion maps. Finally, using the perturbed surgery formula, we compute the Floer homology of their surgeries and the associated d -invariants.

Since $CFL^-(L)$ is the same as $A_{+\infty,+\infty}^-(L)$ viewed as a filtered chain complex (with the Alexander filtration), Corollary 1.2 means the filtered homotopy type of $CFL^-(L)$ contains all the information about the Floer homology of surgeries, when L is a two-bridge link. In [14], it is shown that, for an alternating two-component link L , the filtered chain homotopy type of $\widehat{CFL}(L)$ is determined by the set of data:

- the multi-variable Alexander polynomial $\Delta_L(x, y)$,
- the signature $\sigma(L)$,
- the linking number $\text{lk}(L)$,
- the filtered homotopy type of $\widehat{CFK}(L_i)$ of each component.

However, it is hard to determine the filtered homotopy type of $CFL^-(L)$ in general. For two-bridge links, the nice diagrams show that the U_1, U_2 -differentials in $CFL^-(L)$ are quite simple. In addition, every component of a two-bridge link is the unknot. Thus for two-bridge links, we raise the following question:

Question 1.3. *Given an oriented two-bridge link L , is the filtered homotopy type of $CFL^-(L)$ determined by the set of data $\{\Delta_L(x, y), \sigma(L), \text{lk}(L)\}$?*

We note that \mathbf{HF}^- of surgeries on two-bridge links may also be computed by other methods. For example, as long as one of the framing coefficients is not 0, one can view one component as a knot in a lens space and compute using the methods in [1]. Nevertheless, the method in this paper is more conceptual and fast. Some of the arguments here could be also potentially used for other classes of links.

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2. HEEGAARD DIAGRAMS AND GENERALIZED FLOER COMPLEXES

In this section, we give the precise definitions of what we need in the link surgery formula, including Heegaard diagrams, generalized Floer complexes, polygon maps, and nice diagrams. Here, the Heegaard diagram is adapted for a link inside a 3-manifold with multiple basepoints; the generalized Floer complexes of a link L are derived from the filtered complex $CFL^-(L)$, and they govern the large surgeries; the polygon maps are used in constructing cobordism maps and certain maps in the link surgery formula; knowledge of nice diagrams are also introduced to deal with two-bridge links.

2.1. Heegaard diagrams of links. We give the most general definition of Heegaard diagrams for an oriented link \vec{L} inside a 3-manifold M^3 . When the link $\vec{L} = \emptyset$, the Heegaard diagram is simply for M^3 .

Definition 2.1 (Heegaard diagram of links). A *multi-pointed Heegaard diagram* for the oriented link \vec{L} in M^3 is the data of the form $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$, where:

- Σ is a closed, oriented surface of genus g ;
- $\alpha = \{\alpha_1, \dots, \alpha_{g+k-1}\}$ is a collection of disjoint, simple closed curves on Σ which span a g -dimensional lattice of $H_1(\Sigma; \mathbb{Z})$, hence specify a handlebody U_α ; the same goes for $\beta = \{\beta_1, \dots, \beta_{g+k-1}\}$ specify a handlebody U_β .

- $\mathbf{w} = \{w_1, \dots, w_k\}$ and $\mathbf{z} = \{z_1, \dots, z_m\}$ (with $k \geq m$) are collections of points on Σ with the following property. Let $\{A_i\}_{i=1}^k$ be the connected components of $\Sigma - \alpha_1 - \dots - \alpha_{g+k-1}$ and $\{B_i\}_{i=1}^k$ be the connected components of $\Sigma - \beta_1 - \dots - \beta_{g+k-1}$. Then there is a permutation σ of $\{1, \dots, m\}$ such that $w_i \in A_i \cap B_i$ for $i = 1, \dots, k$, and $z_i \in A_i \cap B_{\sigma(i)}$ for $i = 1, \dots, m$, such that connecting w_i to z_i inside A_i and connecting z_i to $w_{\sigma(i)}$ inside B_i will give rise to the link \vec{L} .

Definition 2.2 (Basic diagrams of links). A Heegaard diagram of \vec{L} is called *basic*, if $l = k = m$, meaning there are exactly two basepoints w_i, z_i for every component \vec{L}_i and no free basepoints.

Definition 2.3 (Admissible diagrams). A *periodic domain* is a two-chain ϕ on Σ which is a linear combination of components of $\Sigma - \alpha \cup \beta$ with integral coefficients, such that the local multiplicity of ϕ at every $w_i \in \mathbf{w}$ is 0. A multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is called *admissible* if every non-trivial periodic domain has some positive local multiplicities and some negative local multiplicities.

Remark 2.4. (1) The definitions of pointed Heegaard moves are systematically formulated in [5] section 4.

(2) In order to avoid the issue of naturality, we fix the Heegaard surface Σ as an embedded surface in the underlying 3-manifold M^3 . Thus, a Heegaard diagram is equivalent to a self-indexed Morse function.

(3) In this paper we will only consider maximally colored diagrams in the sense of [5].

2.2. Generalized Floer complexes. Here we define some chain complexes of a Heegaard diagram for an oriented link in S^3 , which govern the large surgeries on this link. Suppose $\vec{L} = \vec{L}_1 \cup \vec{L}_2 \cup \dots \cup \vec{L}_l$, and \vec{M} is an oriented sublink of \vec{L} , where \vec{M} may not have the induced orientation from \vec{L} on each component. By $\vec{L} - M$, we denote the oriented link obtained by deleting all the components of \vec{M} from \vec{L} .

The identity $H_1(S^3 - \vec{L}) \cong \mathbb{Z}^l$ provides a way to record the Spin^c structures over S^3 relative to L as an affine lattice over \mathbb{Z}^l .

Definition 2.5. Define the affine lattice $\mathbb{H}(\vec{L})$ over $H_1(S^3 - \vec{L})$ as follows:

$$\mathbb{H}(\vec{L})_i = \frac{\text{lk}(\vec{L}_i, \vec{L} - \vec{L}_i)}{2} + \mathbb{Z} \subset \mathbb{Q}, \mathbb{H}(\vec{L}) = \bigoplus_i^l \mathbb{H}(\vec{L})_i,$$

together with its completion

$$\overline{\mathbb{H}}(\vec{L})_i = \mathbb{H}(\vec{L})_i \cup \{-\infty, +\infty\}, \overline{\mathbb{H}}(\vec{L}) = \bigoplus_i^l \overline{\mathbb{H}}(\vec{L})_i.$$

The map $\psi^{\vec{M}} : \mathbb{H}(\vec{L}) \rightarrow \mathbb{H}(\vec{L} - M)$ is defined by $\psi^{\vec{M}}(s) = s - [\vec{M}]/2$. More precisely, let $M = L_{j_1} \cup \dots \cup L_{j_m}$. Then for all i not in $\{j_1, \dots, j_m\}$, let $L_i = (L - M)_{k_i}$, set

$$(2.1) \quad \psi_i^{\vec{M}} : \mathbb{H}(\vec{L})_i \rightarrow \mathbb{H}(\vec{L} - M)_{k_i}, s_i \rightarrow s_i - \frac{\text{lk}(\vec{L}_i, \vec{M})}{2}.$$

The map $\psi^{\vec{M}}$ is defined to be the direct sum of the maps $\psi_i^{\vec{M}}$, for those i 's with L_i not in M .

Given an admissible multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ for \vec{L} , we consider the Lagrangian pair $\mathbb{T}_\alpha, \mathbb{T}_\beta$ in $\text{Sym}^{g+k-1}(\Sigma)$ and the Floer complex $CF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. There is an Alexander multi-grading $A : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \mathbb{H}(L)$ characterized by the property

$$A_i(\mathbf{x}) - A_i(\mathbf{y}) = n_{z_i}(\phi) - n_{w_i}(\phi), \forall \phi \in \pi_2(\mathbf{x}, \mathbf{y})$$

and a normalization condition on the Alexander polynomial. The Alexander grading induces a filtration on $CF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. Given a Spin^c structure on $S^3 - L$, i.e. an element $s \in \mathbb{H}(L)$, we associate a chain complex $\mathfrak{A}^-(\mathcal{H}, s)$ called the *generalized Heegaard Floer complex* using the Alexander filtration. We introduce variables U_i with $1 \leq i \leq l$ for each link component L_i , and U_i with $l+1 \leq i \leq k$ for each free basepoint w_i . For convenience, here we focus on Heegaard diagrams with only one pair of basepoints w_i, z_i on each component and allow free basepoints.

Definition 2.6 (Generalized Floer complex). For $s \in \mathbb{H}(L)$, the *generalized Floer complex* $\mathfrak{A}^-(\mathcal{H}, s)$ is the free module over $\mathcal{R} = \mathbb{F}[[U_1, \dots, U_l]]$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \in \text{Sym}^{g+k-1}(\Sigma)$, and equipped with the differential:

$$(2.2) \quad \partial_s^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}(\alpha) \cap \mathbb{T}(\beta)} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) U_1^{E_{s_1}^1(\phi)} \dots U_l^{E_{s_l}^2(\phi)} U_{l+1}^{n_{w_{l+1}}(\phi)} \dots U_k^{n_{w_k}(\phi)} \cdot \mathbf{y},$$

where $E_s^i(\phi)$ is defined by

$$E_s^i(\phi) = \max\{s - A_i(\mathbf{x}), 0\} - \max\{s - A_i(\mathbf{y}), 0\} + n_{z_i}(\phi) = \max\{A_i(\mathbf{x}) - s, 0\} - \max\{A_i(\mathbf{y}) - s, 0\} + n_{w_i}(\phi).$$

For simplicity, we also write

$$\mathbf{U}^{E_s(\phi)} = \prod_{i=1}^l U_i^{E_{s_i}(\phi)} \prod_{i=l+1}^k U_i^{n_{w_i}(\phi)}.$$

When the Heegaard diagram in the context is unique, we simply denote $\mathfrak{A}^-(\mathcal{H}, s)$ by $A_{s_1, s_2}^-(L)$ or A_{s_1, s_2}^- , where $s = (s_1, s_2)$. The direct product of all the generalized Floer complexes forms the first input of the surgery formula.

Remark 2.7. Let us explain the relation between $A_s^-(L)$ and $CFL^-(L)$. First, if we consider $\mathfrak{A}^-(\mathcal{H}, \infty)$ as a chain complex together with the Alexander filtration, then it is the same as the filtered chain complex $CFL^-(L)$ defined in [14]. Moreover, from Equation (2.2), we see that all the $\mathfrak{A}^-(\mathcal{H}, s)$'s are determined by the filtered complex $\mathfrak{A}^-(\mathcal{H}, \infty)$ which is the same as $CFL^-(L)$.

2.3. Polygon maps and homotopy equivalences between Floer complexes. In the Fukaya category of a symplectic manifold (X, ω) (when it is well-defined), the product of morphisms

$$\mu^2 : \text{Hom}(L_1, L_2) \otimes \text{Hom}(L_0, L_1) \rightarrow \text{Hom}(L_0, L_1)$$

is defined by counting holomorphic triangles. In general, higher products are defined by means of holomorphic polygons. In Heegaard Floer theory, the polygon maps are defined similarly. However, the technical issue is the compactness of moduli spaces of holomorphic polygons in the symmetric product of the Heegaard surface, i.e. whether the polygon counts are finite. This problem breaks down to periodic domains on the Heegaard surface. Admissibility of Heegaard multi-diagrams solves this problem. For the definition of periodic domains and admissibility for multi-diagrams, one can see Definition 4.4 in [5].

Definition 2.8. (1) If two Heegaard diagrams \mathcal{H} and \mathcal{H}' have the same underlying Heegaard surface Σ , and their collections of curves β and β' are related by isotopies and handleslides only (supported away from the basepoints), we say that β and β' are *strongly equivalent*.

(2) Two multi-pointed Heegaard diagrams $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}, \tau)$, $\mathcal{H}' = (\Sigma', \alpha', \beta', \mathbf{w}', \mathbf{z}', \tau')$ are called strongly equivalent, if $\Sigma = \Sigma'$, $\mathbf{w} = \mathbf{w}'$, $\mathbf{z} = \mathbf{z}'$, $\tau = \tau'$, the curve collections α and α' are strongly equivalent, and β and β' are strongly equivalent as well.

(3) We say that two Heegaard diagrams \mathcal{H} and \mathcal{H}' differ by a *surface isotopy* if there is a self-diffeomorphism $\phi : \Sigma \rightarrow \Sigma$ isotopic to the identity and supported away from the link \vec{L} , such that $\Sigma = \Sigma'$ and ϕ takes all the attaching curves and basepoints on Σ to the corresponding one on Σ' . If \mathcal{H} and \mathcal{H}' are surface isotopic, we write $\mathcal{H} \cong \mathcal{H}'$.

Definition 2.9 (Triangle maps). Let $(\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ be a generic, admissible Heegaard triple-diagram, where β and γ are strongly equivalent, such that $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$, $\mathcal{H}' = (\Sigma, \alpha, \gamma, \mathbf{w}, \mathbf{z})$ are both Heegaard diagrams of the link \vec{L} and $\mathcal{H}'' = (\Sigma, \beta, \gamma, \mathbf{w}, \mathbf{z})$ is the Heegaard diagram of the unlink in $\#^{g+1}(S^1 \times S^2)$. Then we can define the triangle map

$$f_{\alpha\beta\gamma} : \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta, s) \otimes \mathfrak{A}^-(\mathbb{T}_\beta, \mathbb{T}_\gamma, s') \rightarrow \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\gamma, s+s')$$

$$f_{\alpha\beta\gamma}(\mathbf{x} \otimes \mathbf{y}) = \sum_{\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) \mid \mu(\phi)=0\}} \#(M(\phi)) \cdot \mathbf{U}^{E_{s,s'}(\phi)} \mathbf{z},$$

where

$$\mathbf{U}^{E_{s,s'}(\phi)} = U_1^{E_{s_1,s'_1}(\phi)} \dots U_l^{E_{s_l,s'_l}(\phi)} U_{l+1}^{n_{w_{l+1}}(\phi)} \dots U_k^{n_{w_k}(\phi)}, s = (s_1, \dots, s_l), s' = (s'_1, \dots, s'_l),$$

$$E_{s,s'}^i(\phi) = \max\{A_i(\mathbf{x}) - s, 0\} + \max\{A_i(\mathbf{y}) - s', 0\} - \max\{A_i(\mathbf{z}) - s - s', 0\} + n_{w_i}(\phi).$$

Definition 2.10 (Quadrilateral maps). Let $(\Sigma, \eta^0, \eta^1, \eta^2, \eta^3, \mathbf{w}, \mathbf{z})$ be a generic, admissible multi-diagram, such that there are two equivalence classes of strongly equivalent attaching curves among $\{\eta^i\}_i$, and η^0, η^3 are in different equivalent classes so that $(\Sigma, \eta^0, \eta^3, \mathbf{w}, \mathbf{z})$ is a Heegaard diagram for the link \vec{L} . Now we can define the quadrilateral maps

$$f_{\eta^0, \dots, \eta^3} : \bigotimes_{i=1}^3 \mathfrak{A}^-(\mathbb{T}_{\eta^{i-1}}, \mathbb{T}_{\eta^i}, s_i) \rightarrow \mathfrak{A}^-(\mathbb{T}_{\eta^0}, \mathbb{T}_{\eta^3}, s_1 + s_2 + s_3)$$

$$f_{\eta^0, \dots, \eta^3}(\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3) = \sum_{\mathbf{y} \in \mathbb{T}_{\eta^0} \cap \mathbb{T}_{\eta^3}} \sum_{\{\phi \in \pi_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}) \mid \mu(\phi)=-1\}} \#(M(\phi)) \cdot \mathbf{U}^{E_{s_1, s_2, s_3}(\phi)} \mathbf{y},$$

where

$$\mathbf{U}^{E_{s_1, s_2, s_3}(\phi)} = U_1^{E_{s_1^1, s_2^1, s_3^1}(\phi)} \dots U_l^{E_{s_1^l, s_2^l, s_3^l}(\phi)} U_{l+1}^{n_{w_{l+1}}(\phi)} \dots U_k^{n_{w_k}(\phi)}, s_i = (s_i^1, \dots, s_i^l), i = 1, 2, 3,$$

$$E_{s_1, s_2, s_3}^i(\phi) = \max\{A_i(\mathbf{x}_1) - s_1, 0\} + \max\{A_i(\mathbf{x}_2) - s_2, 0\} \\ + \max\{A_i(\mathbf{x}_3) - s_3, 0\} - \max\{A_i(\mathbf{y}) - s_1 - s_2 - s_3, 0\} + n_{w_i}(\phi).$$

One can define higher polygon counts $f_{\eta^0 \dots \eta^l}$ similarly, although the case $l > 3$ will not be needed in this paper. For simplicity, we ignore the subscripts of $f_{\eta^0 \dots \eta^l}$. An important property of polygon maps is the so-called *quadratic A_∞ -associativity equation*

$$(2.3) \quad \sum_{0 \leq i < j \leq l} f(x_1, \dots, x_i, f(x_{i+1}, \dots, x_j), x_{j+1}, \dots, x_l) = 0.$$

2.4. Nice diagrams. In [17], Sarkar and Wang use nice Heegaard diagrams to combinatorially compute the $\widehat{HF}(M)$. This algorithm is based on a fact: in a nice diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w})$, the index-1 pseudo-holomorphic disks in $\text{Sym}^{g+k-1}(\Sigma)$ with $n_w = 0$ have simple domains on Σ and can be combinatorially counted.

Definition 2.11 (Nice diagrams). A Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w})$ is called *nice*, if any region (connect components of $\Sigma - \alpha - \beta$) without basepoint $w_i \in \mathbf{w}$ is either a bigon or a square. For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a domain $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is called an *empty embedded $2n$ -gon*, if it is an embedded disk with $2n$ vertices on its boundary, such that for each vertex v , $\mu_v(\phi) = \frac{1}{4}$, and it does not contain any x_i or y_i in its interior. An empty embedded 4-gon is also called empty embedded square.

Remark 2.12. The notation $\pi_2(\mathbf{x}, \mathbf{y})$ in [17] denotes the sets of domains, namely 2-chains ϕ on Σ such that $\partial(\partial\phi|_\alpha) = \mathbf{y} - \mathbf{x}$, whereas in this paper $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the homology classes of Whitney disks in $\text{Sym}^{g+k-1}(\Sigma)$ from \mathbf{x} to \mathbf{y} .

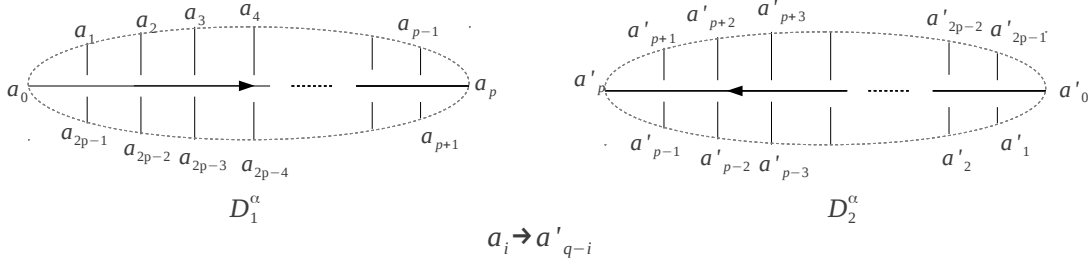


FIGURE 3.1. The Schubert form: the neighborhoods of the two over-bridges.

Theorem 2.13. ([17]) *Let $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ be a domain such that $\mu(\phi) = 1$ and $n_{w_i}(\phi) = 0, \forall i$. If ϕ has a holomorphic representative, then ϕ is either an empty embedded bigon or an empty embedded square. Conversely, if $\phi \in \pi_2(x, y)$ with $n_{w_i}(\phi) = 0, \forall i$ is an empty embedded bigon or an empty embedded square, then the product complex structure on $\Sigma \times D^2$ achieves transversality for ϕ under a generic perturbation of the α and the β curves, and $\mu(\phi) = 1$ as well as $\#M(\phi)/\mathbb{R} = 1 \pmod{2}$.*

The above theorem enables us to combinatorially count differentials in \widehat{CF} on a nice diagram, by counting empty embedded bigons and squares. Following the same lines of the proof, we can obtain the following adaption.

Proposition 2.14. *Suppose $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a Heegaard diagram such that any region of Σ is either a bigon or a square. Then, there is a 1-1 correspondence between the differentials in $\mathfrak{A}^-(\mathcal{H}, \infty)$ and the set of empty embedded bigons and empty embedded squares. Thus, the complex $CF^-(\mathcal{H}, s)$ can be described combinatorially.*

3. GENERALIZED FLOER COMPLEXES OF TWO-BRIDGE LINKS

In this section, we combinatorially compute the generalized Floer complexes $A_{s_1, s_2}^-(\vec{L})$ for all two-bridge links \vec{L} by using nice diagrams.

3.1. Schubert normal form. A two-bridge link/knot can be obtained by closing a rational tangle. For the definition of rational tangles, one can see the reference [8] chapter 9 and [2] chapter 7E, 12D. Let us adopt the notations in [2]. By $b(p, q)$ where $\gcd(p, q) = 1$, we denote the two-bridge link/knot according to the rational tangle of slope $\frac{q}{p}$.

Definition 3.1 (Schubert normal form). For a two-bridge link/knot $L = b(p, q)$, the *Schubert normal form* is a canonical projection of L with two over-bridges and two under-bridges, where we regard the projection plane as a sphere S in S^3 . The two over-bridges O_1, O_2 are straight segments on the projection plane, and each component of the other two under-bridges U_1, U_2 crosses O_1, O_2 alternatively. Together with the lower half space (which is a ball in S^3), the under-bridges U_1, U_2 form a rational tangle of slope $\frac{q}{p}$. Moreover, if L has two components, we arrange the notation such that $L_i = O_i \cup U_i$. We denote the Schubert form by (S, O_1, O_2, U_1, U_2) .

Concretely, the Schubert normal form can be obtained by gluing two disks D_1^α, D_2^α shown in Figure 3.1. The endpoint a_i is glued to a'_{q-i} , where all the subscripts are modulo $2p$. When L has two components, L can be endowed with a canonical orientation induced by the orientation of $\vec{O}_1 = \overrightarrow{a_0 a_p}, \vec{O}_2 = \overrightarrow{a'_0 a'_p}$, which is also shown in Figure 3.1.

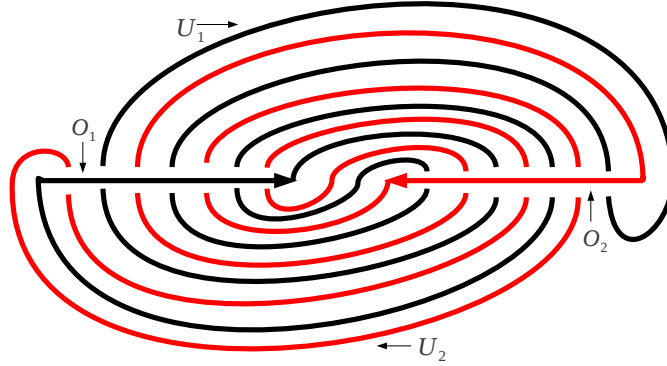


FIGURE 3.2. The Schubert normal form of the Whitehead link.

Example 3.2. In Figure 3.2, we show the Schubert normal form of the Whitehead link $b(8, 3)$.

Remark 3.3. Here in the definition of Schubert normal form, we set the straight arcs to be over-bridges, while in other places the straight arcs are set to be under-bridges. However, given p, q , these two links are the same up to taking the mirror of each other.

Fact 3.4. Let $b(p, q)$ denote the two-bridge link defined as above, where $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$, $p > 0$. Then

(1)(12.1. in [2]) When p is odd, the link $b(p, q)$ is a knot; and when p is even, $b(p, q)$ has two components.

(2)(Schubert, [2], Theorem 12.6.) As an oriented link/knot, $b(p, q)$ is equivalent to $b(p', q')$ if and only if $p' = p, q' \equiv q^{\pm 1} \pmod{2p}$; as an unoriented link/knot, $b(p, q)$ is equivalent to $b(p', q')$ if and only if $p' = p, q' \equiv q^{\pm 1} \pmod{p}$.

(3)(12.8 in [2]) When $b(p, q)$ has two components, the link $b(p, -q)$ is the mirror of $b(p, q)$, and the link $b(p, q + p)$ can be obtained by changing the orientation on one component of $b(p, q)$.

(4)(Remark 12.7, [2]) The linking number can be computed by the formula:

$$\text{lk}(b(p, q)) = \sum_{i=1}^{\frac{p}{2}} (-1)^{\lfloor \frac{(2i-1)q}{p} \rfloor}.$$

(5)(Theorem 9.3.6, [8]) The signature can be computed by the formula:

$$\sigma(b(p, q)) = \sum_{i=1}^{p-1} (-1)^{\lfloor \frac{iq}{p} \rfloor}.$$

3.2. Heegaard diagrams of two-bridge links. In this section, we construct nice Heegaard diagrams of two-bridge links by using their Schubert forms.

Definition 3.5. A *bridge presentation* of a link L is a topological pair (L, S) inside S^3 , such that

- S is an embedded sphere transversely intersecting L ,
- $S^3 - S \cong B_1 \cup B_2$, where B_1, B_2 are homeomorphic to the unit ball B^3 ,
- each pair $(L \cap B_i, B_i)$ with $i = 1, 2$ is homeomorphic to the pair $(\{P_j\}_{j=1}^k \times I, D^2 \times I)$, where $\{P_j\}_{j=1}^k$ is a set of points in the interior of the unit disk D^2 .

The minimum over all possible k is called the *bridge number* of the link, denoted by $\text{br}(L)$.

Every bridge presentation of L gives rise to a genus-0 multi-pointed Heegaard diagram for L . Let the sphere S be the Heegaard surface. In each ball B_i , choose $k - 1$ disjoint proper disks to

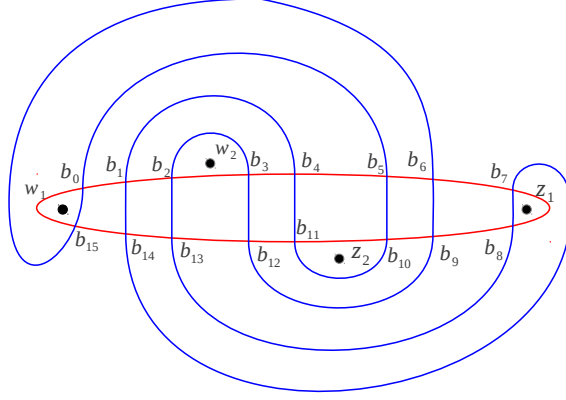


FIGURE 3.3. **The Schubert Heegaard diagram of the Whitehead link.** The red curve is α , and the blue curve is β .

divide B_i into k chambers, such that every component of the k bridges is in a distinct chamber. The boundaries of these disks are the alpha, beta curves. The basepoints w_i, z_i are the intersection points of L and S . In other words, by pushing all the bridges onto the sphere S , we obtain a projection of L consisting of $2k$ arcs $a_1, \dots, a_k, b_1, \dots, b_k$, such that the arcs $\{a_i\}$ are disjoint, the arcs $\{b_j\}$ are disjoint, and the arcs $\{a_i\}$ are always over the arcs $\{b_j\}$. Then the boundaries of tubular neighborhoods of the arcs $\{a_i\}_{i=1}^{k-1}, \{b_j\}_{j=1}^{k-1}$ are the alpha, beta curves.

Definition 3.6 (Schubert Heegaard diagrams). Let $\vec{L} = \vec{L}_1 \cup \vec{L}_2$ be a two-bridge link, and let (S, O_1, O_2, U_1, U_2) be its Schubert form. The *Schubert Heegaard diagram* of L is the Heegaard diagram $\mathcal{H} = (S^2, \{\alpha\}, \{\beta\}, \{z_1, z_2\}, \{w_1, w_2\})$, where

- $\alpha = \partial N(O_1), \beta = \partial N(U_1)$ with $N(O_1), N(U_1)$ being disjoint tubular neighborhoods of O_1, U_1 on S ,
- $\{z_1, w_1\} = \{L_1 \cap S\}$ and $\{z_2, w_2\} = \{L_2 \cap S\}$.

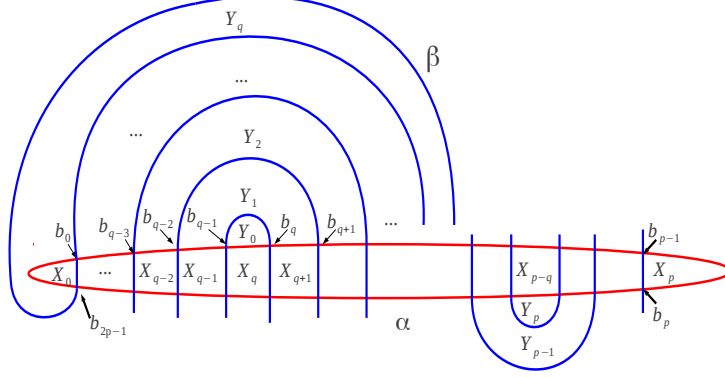
Concretely, regarding the Schubert form as the gluing of two disks D_1^α, D_2^α in Figure 3.1, we can take $\alpha = \partial D_1^\alpha$ and $\beta = \partial N(U_1)$. The basepoint z_1 can be any point in D_1^α near a_p , and the basepoint w_1 can be any point in D_1^α near a_0 ; whereas the basepoint z_2 can be any point in D_2^α near a'_p , and the basepoint w_2 can be any point in D_2^α near a'_0 .

Example 3.7. The two-bridge link $b(8, 3)$ is the Whitehead link Wh (or its mirror due to the convention). The Schubert Heegaard diagram of Wh is in Figure 3.3.

Notations 3.8. Since we will repeatedly discuss the Schubert Heegaard diagram, it is convenient to make a notational convention for all the intersection points and regions as follows.

- The components of $S - \beta$ are both disks, denoted by D_1^β, D_2^β such that the disk D_i^β is a neighborhood of U_i .
- There is a total of four bigons among the components of $S - (\alpha \cup \beta)$, and each of them contains a distinct basepoint in $\{z_1, z_2, w_1, w_2\}$. All the other components are squares.
- We label all the $p + 1$ components of $D_1^\alpha - \beta$ by X_0, X_1, \dots, X_p , and label all the $p + 1$ components of $D_2^\alpha - \beta$ by Y_0, Y_1, \dots, Y_p , such that X_0, X_p, Y_0 , and Y_p are bigons and $w_1 \in X_0, w_2 \in Y_0, z_1 \in X_p, z_2 \in Y_p$.
- There is a total of $2p$ intersection points of α and β . We label them by $b_0, b_1, \dots, b_{2p-1}$ clockwise, such that b_0, b_{2p-1} are vertices of X_0 , and b_{p-1}, b_p are vertices of X_p . All the subscripts are modulo $2p$.

The above properties and conventions are illustrated in Figure 3.4.

FIGURE 3.4. The Schubert Heegaard diagram for the two-bridge link $b(p, q)$.

Lemma 3.9. *The Schubert Heegaard diagram of the two-bridge link $b(p, q)$ is a nice diagram. By Proposition 2.14, the generalized Floer complexes of the Schubert Heegaard diagram are combinatorial.*

From the property of Schubert normal form, it follows a direct description of the Schubert Heegaard diagram.

Lemma 3.10. *In the Schubert Heegaard diagram of $b(p, q)$,*

- (1) *in the disk D_1^α , the points b_i and b_{2p-1-i} are connected by a β -arc,*
- (2) *in the disk D_2^α , the points b_i and b_j are connected by a β -arc if and only if $i + j \equiv 2q - 1 \pmod{2p}$.*

3.3. The multi-variable Alexander polynomial of two-bridge links. With the help of link Floer homology, we can directly calculate the multi-variable Alexander polynomial of knots and links. In [14], there is a formula of the Euler characteristic of $\widehat{HFL}(L)$:

$$(3.1) \quad \sum_{h \in \mathbb{H}(L)} \chi(\widehat{HFL}_*(L, h)) \cdot e^h = \prod_{i=1}^l (T_i^{\frac{1}{2}} - T_i^{-\frac{1}{2}}) \Delta_L.$$

Definition 3.11 (Thin complex and E_2 -collapsed complex). Suppose (C, ∂) is a \mathbb{Z}^2 -filtered chain complex of \mathbb{F} -vector spaces. Let (i, j) denote the filtration, and let g denote the internal grading. The complex (C, ∂) is called *thin*, if $i + j - g$ is a constant for all elements in C . The chain complex C is called *E_2 -collapsed*, if the differential can be decomposed as $\partial = \partial_1 + \partial_2$, such that $F(\partial_1(x)) = F(x) - (1, 0)$ and $F(\partial_2(x)) = F(x) - (0, 1)$, where $F(x)$ is the \mathbb{Z}^2 -filtration of x .

Remark 3.12. A thin complex is E_2 -collapsed. The classification of E_2 -collapsed complexes of \mathbb{F} -vector spaces is shown in [14] Section 12.1.

Proposition 3.13. *Let $L = b(p, q)$ be a two-bridge link, where p is even and $-p < q < p$. Let $A(b_i)$ be the Alexander grading, and let q^{-1} be the number theoretical reciprocal of q modulo $2p$. Then*

$$A(b_i) - A(b_{i-1}) = \begin{cases} ((-1)^{\lfloor q^{-1} \cdot i/p \rfloor}, 0), & i \text{ is even,} \\ (0, (-1)^{\lfloor (q^{-1} \cdot i + 1)/p \rfloor}), & i \text{ is odd.} \end{cases}$$

Furthermore, we have $A_1(b_i) + A_2(b_i) - M(b_i)$ is a constant, i.e. not dependent on i , where $M(b_i)$ is the Maslov grading. In other words, the chain complex $A_{+\infty, +\infty}^-(L)$ is thin.

Proof. We use 3.8. There is a set of bigons of Maslov index 1 connecting b_i and b_{i+1} , for $i = 0, 1, \dots, 2p - 2$. Each of these bigons is a part of one of the disks D_1^β and D_2^β . In fact, these bigons can be obtained by chasing the under-bridges U_1 and U_2 .

The under-bridge U_1 starts from $z_1 = a_p = a'_{q-p}$ and passes the disks D_2^α and D_1^α alternately. For i even, at the point a_i , if the under-bridge U_1 is pointing out of D_1^α , then $i \equiv p - 2kq \pmod{2p}$ for some k with $0 < k < \frac{p}{2}$, which is equivalent to $\lfloor \frac{i \cdot q^{-1}}{p} \rfloor$ is even. In this case, there is a bigon ϕ from b_i to b_{i-1} with a single basepoint z_1 on it, and thereby

$$A(b_i) - A(b_{i-1}) = ((-1)^{\lfloor \frac{q^{-1} \cdot i}{p} \rfloor}, 0).$$

If the under-bridge U_1 is pointing into D_1^α , then $i \equiv 2kq - p \pmod{2p}$ for some k with $0 < k < \frac{p}{2}$, which is equivalent to $\lfloor \frac{i \cdot q^{-1}}{p} \rfloor$ is odd. In this case, there is a bigon ϕ from b_{i-1} to b_i with a single basepoint z_1 on it, and still

$$A(b_i) - A(b_{i-1}) = ((-1)^{\lfloor \frac{q^{-1} \cdot i}{p} \rfloor}, 0).$$

Similarly, by keeping track of U_2 , we can prove the other cases. For i odd, at the point a_i , if the under-bridge U_2 is pointing off D_1^α , then $i \equiv p - q - 2kq \pmod{2p}$ for some k with $0 < k < \frac{p}{2}$, which is equivalent to $\lfloor \frac{1+i \cdot q^{-1}}{p} \rfloor$ is even. In this case, there is a bigon ϕ of index 1 from b_i to b_{i-1} with a single basepoint z_2 on it. Thus, we have $A(b_i) - A(b_{i-1}) = (0, (-1)^{\lfloor \frac{q^{-1} \cdot i + 1}{p} \rfloor})$ for i odd.

From Lipshitz's formula $\mu(\phi) = e(\phi) + \mu_{b_i}(\phi) + \mu_{b_{i-1}}(\phi)$, it follows $\mu(\phi) = 1$, and thereby for all i ,

$$A_1(b_i) + A_2(b_i) - M(b_i) = A_1(b_{i-1}) + A_2(b_{i-1}) - M(b_{i-1}).$$

□

Now we are able to compute the multi-variable Alexander polynomial of two-bridge links by Equation (3.1). Since the Floer chain complex for the Schubert Heegaard diagram is thin, there are no differentials in the associated graded complex of $\widehat{CFL}(L, h)$ for the Alexander filtration. That is, for this thin complex, $\widehat{HFL}(L, h) = \widehat{CFL}(L, h)$. Thus,

$$\prod_{i=1}^l (T_i^{\frac{1}{2}} - T_i^{-\frac{1}{2}}) \cdot \Delta_L(x, y) = \sum_{i=0}^{2p-1} (-1)^{A_1(b_i) + A_2(b_i)} \cdot x^{A_1(b_i)} \cdot y^{A_2(b_i)}.$$

By computer experiments, we have found two distinct two-bridge links $b(126, 47)$ and $b(126, 55)$ that share the same Alexander polynomial, signature, and linking number, but are not the same or mirror to each other.

$$\begin{aligned} \Delta_{b(126, 47)}(x, y) &= \Delta_{b(126, 55)}(x, y) = -15 + \frac{8}{x} + 8x + \frac{y}{8} + 8y - \frac{4}{xy} - 4xy - \frac{4x}{y} - \frac{4y}{x}, \\ \sigma(b(126, 47)) &= \sigma(b(126, 55)) = 3, \\ \text{lk}(126, 47) &= \text{lk}(126, 55) = 1. \end{aligned}$$

3.4. The Floer complexes for two-bridge links. Let $\mathcal{H} = (S, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be the Schubert Heegaard diagram of $b(p, q)$. By Lemma 3.9, the generalized Floer complex $\mathfrak{A}^-(\mathcal{H}, s)$ is combinatorial. It consists of counting the empty embedded bigons of Maslov index 1 on S , since here $g + k - 1 = 1$.

The pattern of empty embedded bigons of Maslov index 1 is illustrated in Figure 3.5 as in [17]. In Schubert Heegaard diagram, these bigons are always in a similar form of Figure 3.5, where the function f_p is defined as follows.

Definition 3.14. For all $n, m, k \in \mathbb{Z}$, let $\text{Mod}(n, m, k)$ be the residue of n modulo m starting from k , that is,

$$\text{Mod}(n, m, k) \equiv n \pmod{m} \quad \text{and} \quad k \leq \text{Mod}(n, m, k) \leq k + m - 1.$$

Then f_p is defined by

$$f_p(n) = |\text{Mod}(n, 2p, -p + 1)|.$$

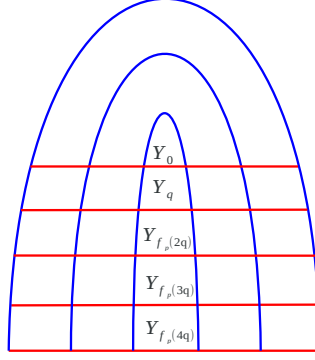


FIGURE 3.5. **A bigon in the Schubert Heegaard diagram of a two-bridge link.** The red lines are parts of α , and the blue curves are parts of β .

Lemma 3.15. *In the Schubert Heegaard diagram $(S, \{\alpha\}, \{\beta\}, \{w_1, w_2\}, \{z_1, z_2\})$ of the two-bridge link $\vec{L} = b(p, q)$, the regions in D_1^β are $X_0, Y_q, X_{f_p(2q)}, Y_{f_p(3q)}, \dots, X_{f_p(pq)} = X_p$ consecutively, and the regions in D_2^β are $Y_0, X_q, Y_{f_p(2q)}, X_{f_p(3q)}, \dots, Y_{f_p(pq)} = Y_p$ consecutively.*

Proof. Note that D_i^β is the regular neighborhood of the under-bridge U_i . The region X_i contains the arc $a_i a_{-i} \subset L$, and the region Y_j contains arc $a'_j a'_{-j} \subset L$. Conversely, the point a_i is contained in $X_{f_p(i)}$, and the point a'_j is contained in $Y_{f_p(j)}$. Thus since a_i, a'_{q-i} are glued together and a_{-i}, a'_{q+i} are glued together, $X_{f_p(i)}$ is adjacent to $Y_{f_p(q-i)}$ and $Y_{f_p(q+i)}$. Since Y_0 is in D_2^β and it is adjacent to X_q , the region X_q is adjacent to $Y_{f_p(2q)}$. Inductively, we can show in D_2^β , $Y_{f_p(kq)}$ is adjacent to $X_{f_p[(k-1)q]}$ and $X_{f_p[(k+1)q]}$. A similar argument applies to D_1^β . \square

Definition 3.16. In the bigon ϕ , denote the number of α arcs in ϕ by $n_\alpha(\phi)$, and denote the number β arcs in ϕ by $n_\beta(\phi)$.

Every bigon ϕ is uniquely determined by $n_\alpha(\phi), n_\beta(\phi)$ and the basepoint on it.

Lemma 3.17 (Patterns of bigons). *In the Schubert Heegaard diagram of $b(p, q)$, suppose ϕ is an empty embedded bigon of index 1 in $\pi_2(b_i, b_j)$. Then*

$$\begin{aligned} (i, j) &= ((1 - n_\alpha)q + n_\beta - 1, (1 - n_\alpha)q - n_\beta), & \text{if } w_1 \in \phi, n_\alpha \text{ is odd,} \\ (i, j) &= (n_\alpha q - n_\beta, n_\alpha q + n_\beta - 1), & \text{if } w_1 \in \phi, n_\alpha \text{ is even,} \\ (i, j) &= (n_\alpha q - n_\beta, n_\alpha q + n_\beta - 1), & \text{if } w_2 \in \phi, n_\alpha \text{ is odd,} \\ (i, j) &= ((1 - n_\alpha)q + n_\beta - 1, (1 - n_\alpha)q - n_\beta), & \text{if } w_2 \in \phi, n_\alpha \text{ is even.} \end{aligned}$$

Furthermore, given $m, n \in \mathbb{Z}$ and a basepoint $pt \in \{w_1, w_2, z_1, z_2\}$, there exists at most one empty embedded bigon ϕ with $n_\alpha(\phi) = m$, $n_\beta(\phi) = n$, and $n_{pt}(\phi) = 1$.

There exists an empty embedded bigon ϕ of index 1 with $n_\alpha(\phi) = m$, $n_\beta(\phi) = n$, and $n_{w_2}(\phi) = 1$ if and only if the condition $P_1(m, n)$ holds.

The condition $P_1(m, n)$ is as follows:

- (1) either $m = 1$, or if $m > 1$, then the set of intervals: $[0, n - 1]$ and all intervals $[f_p(2iq) - n + 1, f_p(2iq) + n - 1]$ with $1 \leq 2i \leq m - 1$ are pairwise disjoint intervals in $[0, p]$;
- (2) either $m = 1$, or if $m > 1$, then the set of intervals: all intervals $[f_p((2i + 1)q) - n + 1, f_p((2i + 1)q) + n - 1]$ with $1 \leq 2i + 1 \leq m - 1$ are also pairwise disjoint intervals in $[1, p - 1]$.

Similarly, there exists an empty embedded bigon ϕ of index 1 with $n_\alpha(\phi) = m$, $n_\beta(\phi) = n$ and $n_{w_1}(\phi) = 1$ if and only if the condition $P_2(m, n)$ holds.

The condition $P_2(m, n)$ is as follows:

- (1) either $n = 1$, or $n > 1$ and the set of intervals: $[0, m - 1]$ and all intervals $[f_p(2iq) - m + 1, f_p(2iq) + m - 1]$ with $1 \leq 2i \leq n - 1$ are pairwise disjoint intervals in $[0, p]$;
- (2) either $n = 1$, or $n > 1$ and the set of intervals: all intervals $[f_p((2i + 1)q) - m + 1, f_p((2i + 1)q) + m - 1]$ with $1 \leq 2i + 1 \leq n - 1$ are also pairwise disjoint intervals in $[1, p - 1]$.

In addition, for $i = 1, 2$, there is a one-to-one correspondence between the set of all the empty embedded bigons with $n_{w_i} = 1$ and the set of empty embedded bigons with $n_{z_i} = 1$, where the bigon $\phi \in \pi_2(b_i, b_j)$ with $n_\alpha = m, n_\beta = n, n_{w_i} = 1$ is sent to the bigon $\phi' \in \pi_2(b_i + p, b_j + p)$ with $n_\alpha = m, n_\beta = n, n_{z_i} = 1$.

Proof. Suppose ϕ is a bigon of index 1 in $\pi_2(b_i, b_j)$ with $n_{w_2}(\phi) = 1$. Combining Lemma 3.10 and Lemma 3.15, we can get the formula of (i, j) out of Figure 3.5 by induction on n_α, n_β . The initial step is $(i, j) = (q - 1, q)$, for $n_\alpha = n_\beta = 1$. Similarly, we can show the other case where $n_{w_1}(\phi) = 1$.

For the second part, the sufficient and necessary condition of when there exists an empty embedded bigon is that all the regions in the bigon are not overlapped. By Lemma 3.15, it is not hard to get the formulas by induction.

Finally, notice that there is a symmetry of the Heegaard Schubert diagram which sends b_k to b_{k+p} and exchanges w_i to z_i for all $1 \leq k \leq 2p, i = 1, 2$. This symmetry directly gives the one-to-one correspondence between the bigons with w_i and the ones with z_i . \square

Consequently, we get an algorithm for computing $A_s^-(b(p, q))$ as follows.

Theorem 3.18. *Let $L = b(p, q)$ be a two-bridge link and \mathcal{H} be the Schubert Heegaard diagram. Define functions $F_i : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}/2p\mathbb{Z} \times \mathbb{Z}/2p\mathbb{Z}, i = 1, 2$, by*

$$F_1(m, n) = \begin{cases} ((1 - m)q + n - 1, (1 - m)q - n), & \text{if } m \text{ is odd,} \\ (mq - n, mq + n - 1), & \text{if } m \text{ is even.} \end{cases}$$

$$F_2(m, n) = \begin{cases} ((1 - m)q + n - 1, (1 - m)q - n), & \text{if } m \text{ is even,} \\ (mq - n, mq + n - 1), & \text{if } m \text{ is odd.} \end{cases}$$

The conditions $P_i(m, n), i = 1, 2$ are as in Lemma 3.17.

Then, the complex $\mathfrak{A}^-(\mathcal{H}, +\infty, +\infty)$ is a free $\mathbb{F}[[U_1, U_2]]$ -module generated by g_0, \dots, g_{2p-1} with differentials

$$\partial g_i = \sum_{j=0}^{2p-1} (\lambda_{i+p, j+p} + \mu_{i+p, j+p}) g_j + \sum_{j=0}^{2p-1} \lambda_{i, j} U_1 g_j + \sum_{j=0}^{2p-1} \mu_{i, j} U_2 g_j,$$

where the coefficients $\lambda_{i, j}, \mu_{i, j} \in \mathbb{Z}/2\mathbb{Z}$ are determined by the following equations

$$\lambda_{i, j} = \#\{(m, n) \in \mathbb{N} \times \mathbb{N} | 1 \leq m, n \leq p, F_1(m, n) = (i, j), P_1(m, n) \text{ is true}\} \pmod{2},$$

$$\mu_{i, j} = \#\{(m, n) \in \mathbb{N} \times \mathbb{N} | 1 \leq m, n \leq p, F_2(m, n) = (i, j), P_2(m, n) \text{ is true}\} \pmod{2}.$$

Remark 3.19. One can get an algorithm of $O(p^2)$ time complexity for computing $A_s^-(b(p, q))$. To compute $A_s^-(b(p, q))$, we only need to know $A_{+\infty, +\infty}^-(b(p, q))$, which is determined by all the $\lambda_{i, j}$'s and $\mu_{i, j}$'s. Computing $\lambda_{i, j}$'s and computing $\mu_{i, j}$'s are similar. In order to get all the $\lambda_{i, j}$'s, one can nest two loops. The outer loop is indexed by $n \geq 1$, and the inner loop is indexed by $m \geq 1$ with a test condition $P_1(m, n)$. When $P_1(m, n)$ is true, we change the value of $\lambda_{F_1(m, n)}$ by 1 (mod 2) and keep running the inner loop; when $P_1(m, n)$ is false, we stop the inner loop and go back to the outer loop.

Let us estimate the time complexity. First, when $P_1(m, n)$ is true and $n > q$, m must be 1, as otherwise the second part of $P_1(m, n)$ would imply $f_p(q) - n + 1 = q - n + 1 \geq 1$. Thus we force the outer loop stop when $n = q + 1$. Computing the other $\lambda_{F_1(m, n)}$'s with $(m, n) = (1, n), n > p$ can be done within $O(p)$ operations. Second, if $P_1(m, n)$ is true, then $m \leq (p + 1)/n$. This is because the first part of $P_1(m, n)$ implies that there are m pieces of open intervals $(f_p(2iq) - n + \frac{1}{2}, f_p(2iq) + n - \frac{1}{2})$

pairwise disjoint in $(-\frac{1}{2}, p + \frac{1}{2})$. Thus, at the n^{th} step of the outer loop, the inner loop stops within $\lfloor (p+1)/n \rfloor$ steps. Finally, testing $P_1(m, n)$ can be done within $2m$ steps. In fact, when m is even, we can check if the new interval $[f_p(mq) - n + 1, f_p(mq) + n - 1]$ is disjoint from the other $\frac{m}{2} - 1$ intervals in the first part of $P_1(m, n)$ (which are already ordered in the previous step). If it is disjoint from the other intervals, we put it in the correct position in the order. This is done within $2m$ operations. It is similar when m is odd.

Thus the time complexity is of the order

$$T(p, q) = \left(\sum_{n=1}^q \sum_{m=1}^{\lfloor (p+1)/n \rfloor} 2m \right) + O(p) \leq \left(\sum_{n=1}^q \left[\left(\frac{p+1}{n} \right)^2 + \frac{p+1}{n} \right] \right) + O(p).$$

Since $n \leq q \leq p$, $(p+1)/n \geq 1$, thus

$$T(p, q) \leq 2 \left[\sum_{n=1}^q \frac{(p+1)^2}{n^2} \right] + O(p) = O(p^2).$$

4. LINK SURGERY FORMULA

In this section, we review the link surgery formula of Manolescu-Ozsváth for two-component links with basic diagrams. In Section 4.1, we review some algebra on *hyperboxes of chain complexes* and introduce *twisted gluing* of squares of chain complexes. In Section 4.2, we express the link surgery formula for a two-component link as a twisted gluing of certain squares of chain complexes derived from the link. These squares are elaborated in Section 4.3, by using *primitive systems of hyperboxes*. The primitive systems of hyperboxes are generalizations of the *basic systems of hyperboxes* used in [5]. One can consult [5] for the full generality of link surgery formula with general Heegaard diagrams. We assume that the reader is familiar with Heegaard Floer homology [12, 11, 13, 14].

Throughout, $\vec{L} = \vec{L}_1 \cup \vec{L}_2$ will be an oriented link in S^3 , and \vec{M} will denote an oriented sublink of \vec{L} which may not have the induced orientation from \vec{L} on each component.

4.1. Hyperboxes of chain complexes.

4.1.1. *Hyperboxes of chain complexes.* A hyperbox of chain complexes is a generalization of an iterated mapping cone. A hyperbox of chain complexes can also be regarded as a generalized twisted complexes.

Definition 4.1 (Hyperbox). An n -dimensional *hyperbox* of size $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}_{\geq 0}^n$ is the subset

$$\mathbb{E}(\mathbf{d}) = \{(\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}_{\geq 0}^n : 0 \leq \varepsilon_i \leq d_i\}.$$

If $\mathbb{E}(\mathbf{d}) = \{0, 1\}^n$, then $\mathbb{E}(\mathbf{d})$ is called a *hypercube*, denoted by \mathbb{E}_n .

Definition 4.2 (Hyperbox of chain complexes). Let R be an \mathbb{F} -algebra. An n -dimensional *hyperbox of chain complexes* of size $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$ is a collection of \mathbb{Z} -graded R -modules

$$(C^\varepsilon)_{\varepsilon \in \mathbb{E}(\mathbf{d})}, C^\varepsilon = \bigoplus_{* \in \mathbb{Z}} C_*^\varepsilon,$$

together with a collection of R -linear maps

$$D_{\varepsilon^0}^{\varepsilon} : C_*^{\varepsilon^0} \rightarrow C_{*-1+||\varepsilon||}^{\varepsilon^0 + \varepsilon},$$

one map for each $\varepsilon^0 \in \mathbb{E}(\mathbf{d})$ and $\varepsilon \in \mathbb{E}_n$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\mathbf{d})$. The maps are required to satisfy the relations

$$\sum_{\varepsilon' \leq \varepsilon} D_{\varepsilon^0 + \varepsilon'}^{\varepsilon - \varepsilon'} \circ D_{\varepsilon^0}^{\varepsilon'} = 0,$$

for all $\varepsilon^0 \in \mathbb{E}(\mathbf{d})$, $\varepsilon \in \mathbb{E}_n$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\mathbf{d})$.

By abuse of notation, we sometimes let D^ε stand for any of its map $D_{\varepsilon_0}^\varepsilon$. Note that a hypercube of chain complexes H gives rise to a total complex of the hypercube $\text{Tot}(H)$.

Example 4.3 (1-dimensional hyperboxes). A 1-dimensional hyperbox of chain complexes is a sequence of chain complexes C_n , together with a sequence of chain maps $f_n : C^{(n-1)} \rightarrow C^{(n)}$.

$$C^{(0)} \xrightarrow{f_1} C^{(1)} \xrightarrow{f_2} C^{(2)} \xrightarrow{f_3} \dots \longrightarrow C^{(n-1)} \xrightarrow{f_n} C^{(n)}.$$

The total complex of a 1-dimensional hypercube of chain complexes can be regarded as a mapping cone. Therefore, we also call a 1-dimensional hyperbox of chain complexes a *sequence of chain complexes*.

Example 4.4 (2-dimensional hyperboxes). A *square of chain complexes* is a 2-dimensional hypercube of chain complexes:

$$\begin{array}{ccc} C^{(0,0)} & \xrightarrow{D_{(0,0)}^{(1,0)}} & C^{(1,0)} \\ D_{(0,0)}^{(0,1)} \downarrow & \searrow D_{(0,0)}^{(1,1)} & \downarrow D_{(1,0)}^{(0,1)} \\ C^{(0,1)} & \xrightarrow{D_{(0,1)}^{(1,0)}} & C^{(1,1)} \end{array} = \begin{array}{ccc} C^{(0,0)} & \xrightarrow{D^{(1,0)}} & C^{(1,0)} \\ D^{(0,1)} \downarrow & \searrow D^{(1,1)} & \downarrow D^{(0,1)} \\ C^{(0,1)} & \xrightarrow{D^{(1,0)}} & C^{(1,1)}. \end{array}$$

Here $D^{(1,0)}, D^{(0,1)}$ are chain maps, and $D^{(1,1)}$ is a chain homotopy between $D^{(0,1)} \circ D^{(1,0)}$ and $D^{(1,0)} \circ D^{(0,1)}$. We can regard the total complex of this square as the mapping cone of

$$\text{cone}(D_{(0,0)}^{(1,0)}) \xrightarrow{D_{(0,0)}^{(0,1)} + D_{(1,0)}^{(0,1)} + D_{(0,0)}^{(1,1)}} \text{cone}(D_{(0,1)}^{(1,0)}).$$

A *rectangle of chain complexes* is a 2-dimensional hyperbox of chain complexes. It consists of squares of chain complexes. A rectangle of chain complexes of size $(m, 1)$ can also be regarded as a sequence of mapping cones, i.e. a size (m) 1-dimensional hyperbox of mapping cones.

Let (C, D) be a hyperbox of chain complexes of size (d_1, d_2, \dots, d_n) . Fix $1 \leq i \leq n$, for any integer $0 \leq l \leq d_i$, we have a hyperbox $(C^{\varepsilon_i=l}, D^{\varepsilon_i=l})$ of size $(d_1, \dots, d_{i-1}, 0, d_{i+1}, \dots, d_n)$, where $C^{\varepsilon_i=l}$ consists of $C^{(\varepsilon_1, \dots, \varepsilon_n)}$ with $\varepsilon_i = l$ and the differentials $D^{\varepsilon_i=l}$ consists of all the differentials $D_{\varepsilon_0}^\varepsilon$ of (C, D) inside $C^{\varepsilon_i=l}$.

Remark 4.5. In general, a hyperbox of chain complexes is not a chain complex. But a hypercube is a chain complex considered as the total complex, and it can also be regarded as a mapping cone in many ways.

4.1.2. Compression. From a hyperbox of chain complexes $H = ((C^\varepsilon)_{\varepsilon \in \mathbb{E}(d)}, (D^\varepsilon)_{\varepsilon \in \mathbb{E}_n})$, we can obtain a hypercube of chain complexes $\hat{H} = (\hat{C}^\varepsilon, \hat{D}^\varepsilon)_{\varepsilon \in \mathbb{E}_n}$, thus generating a total complex $\text{Tot}(\hat{H})$. The process of turning H into \hat{H} is called *compression*.

Example 4.6 (Compression of 1-dimensional hyperboxes). Let H be a hyperbox of dimension 1, see Example 4.3. The compression \hat{H} is the mapping cone of the composition of the maps f_1, \dots, f_n

$$C^{(0)} \xrightarrow{f_n \circ \dots \circ f_1} C^{(n)}.$$

Example 4.7 (Compression of 2-dimensional hyperboxes). Consider a rectangle of chain complexes R of size $(n, 1)$:

$$\begin{array}{ccccccc} C^{(0,0)} & \xrightarrow{f_1} & C^{(1,0)} & \xrightarrow{f_2} & C^{(2,0)} & \xrightarrow{f_3} & \dots \xrightarrow{f_n} C^{(n,0)} \\ \downarrow k_0 & \searrow H_1 & \downarrow k_1 & \searrow H_2 & \downarrow k_2 & \searrow H_3 & \dots \searrow H_n \downarrow k_n \\ C^{(0,1)} & \xrightarrow{g_1} & C^{(1,1)} & \xrightarrow{g_2} & C^{(2,1)} & \xrightarrow{g_3} & \dots \xrightarrow{g_n} C^{(n,1)}. \end{array}$$

As in 4.4, we can regard this rectangle as a 1-dimensional hyperbox of mapping cones $\text{cone}(k_i)$, $i = 0, 1, \dots, n$. The compression of 1-dimensional hyperboxes induces the compression of rectangles of chain complexes as follows

$$\begin{array}{ccc} C^{(0,0)} & \xrightarrow{f_n \circ \dots \circ f_1} & C^{(n,0)} \\ k_0 \downarrow & \searrow \hat{H} & \downarrow k_n \\ C^{(0,1)} & \xrightarrow{g_n \circ \dots \circ g_1} & C^{(n,1)}, \end{array}$$

where

$$\hat{H} = \sum_{i=1}^n f_1 \circ \dots \circ f_{i-1} \circ H_i \circ g_{i+1} \circ \dots \circ g_n.$$

Similarly, we can compress a rectangle of chain complexes of size $(1, m)$. For a rectangle of chain complexes R of size (n, m) , we can decompose this rectangle into a union of n vertical rectangles of size $(1, m)$. We first compress all of these n vertical rectangles, and thus get a rectangle R' of size $(n, 1)$. Then we keep compressing R' and get a square of chain complexes \hat{R} . Alternatively, we can also first compress every row, and then compress the column. So every ordering of the coordinate axes gives a different way to compress the rectangle.

For higher dimensional hyperbox, the compression is defined similarly by induction, once we fix an order of the coordinate axes. Let us describe this procedure using the language of composing *chain maps of hyperboxes*. One can check that it is the same as the compression by means of the *algebra of songs* introduced in [5].

Let ${}^0H = (({}^0C^\varepsilon)_{\varepsilon \in \mathbb{E}(\mathbf{d})}, ({}^0D^\varepsilon)_{\varepsilon \in \mathbb{E}_n})$, ${}^1H = (({}^1C^\varepsilon)_{\varepsilon \in \mathbb{E}(\mathbf{d})}, ({}^1D^\varepsilon)_{\varepsilon \in \mathbb{E}_n})$ be two hyperboxes of chain complexes, having the same size $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$. Let $(\mathbf{d}, 1) \in \mathbb{Z}_{\geq 0}^{n+1}$ be the sequence obtained from \mathbf{d} by adding 1 at the end.

Definition 4.8. (Chain maps of hyperboxes) A chain map $F : {}^0H \rightarrow {}^1H$ is a collection of linear maps

$$F_{\varepsilon^0}^\varepsilon : {}^0C_{*}^{\varepsilon^0} \rightarrow {}^1C_{*+\|\varepsilon\|}^{\varepsilon^0+\varepsilon},$$

satisfying

$$\sum_{\varepsilon' \leq \varepsilon} (D_{\varepsilon^0+\varepsilon'}^{\varepsilon-\varepsilon'} \circ F_{\varepsilon^0}^{\varepsilon'} + F_{\varepsilon^0+\varepsilon'}^{\varepsilon-\varepsilon'} \circ D_{\varepsilon^0}^{\varepsilon'}) = 0,$$

for all $\varepsilon^0 \in \mathbb{E}(\mathbf{d})$, $\varepsilon \in \mathbb{E}_n$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\mathbf{d})$.

In other words, a chain map between the hyperboxes 0H and 1H is an $(n+1)$ -dimensional hyperbox of chain complexes, of size $(\mathbf{d}, 1)$, such that the sub-hyperbox corresponding to $\varepsilon_{n+1} = 0$ is 0H and the one corresponding to $\varepsilon_{n+1} = 1$ is 1H . The maps F are those maps D in the new hyperbox that increase ε_{n+1} by 1. Direct computations show the associativity $(F \circ G) \circ H = F \circ (G \circ H)$.

For a n -dimensional hyperbox H of size $\mathbf{d} = (d_1, \dots, d_n)$, we fix an order of the axes, say, the increasing order $1, 2, \dots, n$. There is a hyperbox $C^{\varepsilon_{n+1}=i}$ of size $(d_1, \dots, d_{n-1}, 0)$ which is corresponding to $\varepsilon_{n+1} = i$ for all i with $0 \leq i \leq d_n$. Thus the hyperbox H can be decomposed into d_n pieces of hyperboxes of size $(d_1, \dots, d_{n-1}, 1)$, which is a chain map $F_i : C^{\varepsilon_{n+1}=i-1} \rightarrow C^{\varepsilon_{n+1}=i}$. Thus the composition $F_{d_n} \circ \dots \circ F_1$ is a hyperbox of size $(d_1, \dots, d_{n-1}, 1)$, and we call it the compression along the n^{th} -axis $\text{Comp}_n(H)$. If we keep doing compressions for the other axes, then we get the compression $\hat{H} = \text{Comp}_1 \circ \dots \circ \text{Comp}_n(H)$.

4.1.3. Gluing of squares. In the link surgery formula, an algebraic operation occurs, which we could call a twisted gluing of hypercubes. It consists in repeatedly gluing mapping cones $A \xrightarrow{f} B$, $A \xrightarrow{g} B$ to get a new mapping cone $A \xrightarrow{f+g} B$. In this section we describe this operation in detail for the case of two-component links. We then call it the *twisted gluing of framed product squares*.

Definition 4.9 (Gluing of squares). Suppose there are four squares of chain complexes $R_{i,j} = (C_{i,j}^\varepsilon, D_{i,j}^\varepsilon)$, $i, j = 0, 1$ as listed below,

$$R_{i,j} : \begin{array}{ccc} C_{i,j}^{(0,0)} & \xrightarrow{D_{i,j}^{(1,0)}} & C_{i,j}^{(1,0)} \\ D_{i,j}^{(0,1)} \downarrow & \searrow D_{i,j}^{(1,1)} & \downarrow D_{i,j}^{(0,1)} \\ C_{i,j}^{(0,1)} & \xrightarrow{D_{i,j}^{(1,0)}} & C_{i,j}^{(1,1)}. \end{array}$$

The squares $\{R_{i,j}\}_{i,j}$ are called *gluable*, if $C_{0,0}^\varepsilon = C_{0,1}^\varepsilon = C_{1,0}^\varepsilon = C_{1,1}^\varepsilon$ for all $\varepsilon \in \mathbb{E}_2$ and $D_{i,0}^{(1,0)} = D_{i,1}^{(1,0)} = D_{0,j}^{(0,1)} = D_{1,j}^{(0,1)} = D_j^{(0,1)}$ for all $i, j = 0, 1$. Then we can define $R = (C^\varepsilon, D^\varepsilon)$ to be the *gluing* of $R_{i,j}$'s as below, where we suppress the subscripts i, j of $C_{i,j}^\varepsilon$. One can check that R is a square of chain complexes.

$$R := \begin{array}{ccc} C^{(0,0)} & \xrightarrow{D_0^{(1,0)} + D_1^{(1,0)}} & C^{(1,0)} \\ D_0^{(0,1)} + D_1^{(0,1)} \downarrow & \searrow \sum_{i,j} D_{i,j}^{(1,1)} & \downarrow D_0^{(0,1)} + D_1^{(0,1)} \\ C^{(0,1)} & \xrightarrow{D_0^{(1,0)} + D_1^{(1,0)}} & C^{(1,1)}. \end{array}$$

Definition 4.10 (Framed product square). A \mathbb{Z}^2 -product square of chain complexes is a direct product of squares of chain complexes

$$R = \prod_{s \in \mathbb{Z}^2} R_s,$$

where R_s is a square of chain complexes for all $s \in \mathbb{Z}^2$. We call s the *coordinate* of any element $x \in R_s$. The function

$$\mathcal{F} : R \rightarrow \mathbb{Z}^2, \quad \mathcal{F}(x) = s, \forall s \in R_s,$$

is called the *framing* of R . In order to denote the framing, we write a *framed product square* as a pair

$$(R, \mathcal{F}) = \prod_{s \in \mathbb{Z}^2} (C_s^\varepsilon, D_s^\varepsilon).$$

We can shift the framing \mathcal{F} by a set of vectors in \mathbb{Z}^2 , $\mathbf{V} = \{v^\varepsilon\}_{\varepsilon \in \mathbb{E}_2}$, to get a new framing $\mathcal{F}^\mathbf{V}$, such that

$$\forall x \in C_s^\varepsilon, \quad \mathcal{F}^\mathbf{V}(x) = \mathcal{F}(x) + v^\varepsilon.$$

We call the new framed product square $(R, \mathcal{F}^\mathbf{V})$ the *shifted square of R by \mathbf{V}* , and simply denote it by $R[\mathbf{V}]$. Thus, we can write $R[\mathbf{V}] = \prod_{s \in \mathbb{Z}^2} (\tilde{C}_s^\varepsilon, \tilde{D}_s^\varepsilon)$, where $\tilde{C}_{s+v^\varepsilon}^\varepsilon = C_s^\varepsilon, \forall \varepsilon \in \mathbb{E}_2, \forall s \in \mathbb{Z}^2$.

Definition 4.11 (Framed gluable). Let $(R_{i,j}, \mathcal{F}_{i,j})$ with $i, j = 0, 1$ be a set of framed product squares of chain complexes. The set of four squares $\{R_{i,j}\}_{i,j}$ is called *framed gluable*, if $\{R_{i,j}\}_{i,j}$ are gluable as squares of chain complexes and all the framings $\mathcal{F}_{i,j}, \forall i, j$ are the same. Then, the result is called the *framed gluing* of $(R_{i,j}, \mathcal{F}_{i,j})$'s.

Definition 4.12 (Twisted gluing). Let $(R_{i,j}, \mathcal{F}_{i,j})$ with $i, j = 0, 1$ be a set of framed product squares of chain complexes. For any matrix $\Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{Z}^{2 \times 2}$, let

$$\mathbf{V}_{i,j}(\Lambda) = \{v^\varepsilon = \Lambda \cdot (i\varepsilon_1, j\varepsilon_2)^t\}_{\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{E}_2}, \quad \forall i, j = 0, 1.$$

Then there are four shifted squares $R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]$, with $i, j = 0, 1$. As long as these four shifted squares $\{R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]\}_{i,j=0,1}$ are framed gluable, we define the Λ -twisted gluing of $\{R_{i,j}\}_{i,j}$ to be the framed gluing of $\{R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]\}_{i,j}$, denoted by R^Λ . See Figure 4.1 for an example of twisted gluing.

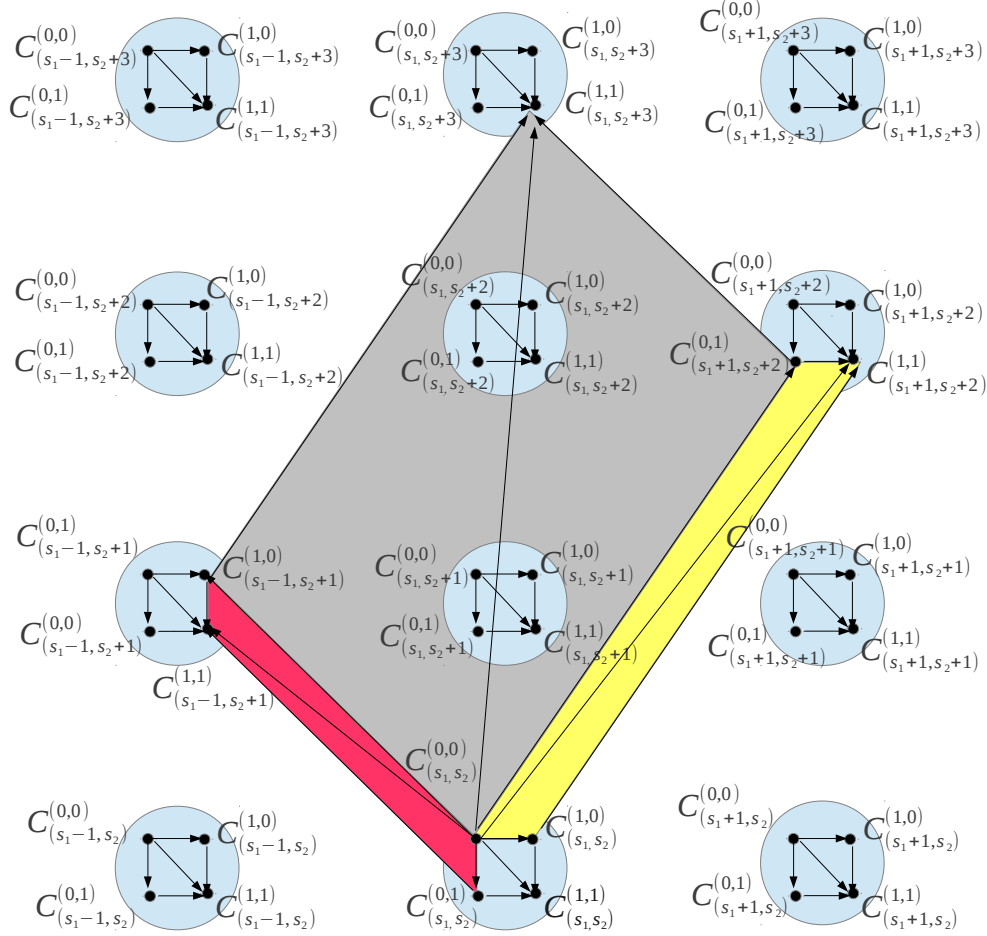


FIGURE 4.1. **An example of twisted gluing.** This is an example of $\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$ -twisted gluing of four squares $\{R_{i,j} = \prod_{s \in \mathbb{Z}^2} R_{s,i,j}\}_{i,j=0,1}$, where $R_{s,i,j} = (C_{s,i,j}^\varepsilon, D_{s,i,j}^\varepsilon)$. Every shaded circle encloses a factor $R_{s,0,0}$ of the \mathbb{Z}^2 -product square $R_{0,0}$ with some $s \in \mathbb{Z}^2$. The yellow parallelogram indicates the D -maps of $R_{0,1}$, which is shifted by the vector Λ_2 ; whereas the red parallelogram indicates the D -maps of $R_{1,0}$, which is shifted by the vector Λ_1 . The gray parallelogram indicates all the maps of the square $R_{1,1}$, which is shifted by using both Λ_1 and Λ_2 .

Example 4.13 (Twisted gluing in link surgery formula). Suppose for any $(i, j) \in \mathbb{E}_2$,

$$R_{i,j} = \prod_{s \in \mathbb{Z}^2} R_{s,i,j}$$

with $R_{s,i,j} = (C_{s,i,j}^\varepsilon, D_{s,i,j}^\varepsilon)$ is a framed product square of chain complexes with the natural framing $\mathcal{F}_{i,j}(x) = s, \forall x \in C_{s,i,j}$. Let \vec{L} be a two-component link and lk be the linking number. Given any surgery framing matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \text{lk} \\ \text{lk} & \lambda_2 \end{pmatrix},$$

as long as the following identities hold for all $s \in \mathbb{Z}^2$,

$$\begin{aligned}
 (4.1) \quad & C_{s,0,0}^{(0,0)} = C_{s,0,1}^{(0,0)} = C_{s,1,0}^{(0,0)} = C_{s,1,1}^{(0,0)}, \\
 & C_{s,0,0}^{(1,0)} = C_{s,0,1}^{(1,0)} = C_{s-\Lambda_1,1,0}^{(1,0)} = C_{s-\Lambda_1,1,1}^{(1,0)}, \\
 & C_{s,0,0}^{(0,1)} = C_{s-\Lambda_2,0,1}^{(0,1)} = C_{s,1,0}^{(0,1)} = C_{s-\Lambda_2,1,1}^{(0,1)}, \\
 & C_{s,0,0}^{(1,1)} = C_{s-\Lambda_2,0,1}^{(1,1)} = C_{s-\Lambda_1,1,0}^{(1,1)} = C_{s-\Lambda_1-\Lambda_2,1,1}^{(1,1)},
 \end{aligned}$$

the shifted squares $\{R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]\}_{i,j}$ are framed gluable. Thus we define the twisted gluing of squares R^Λ .

Remark 4.14. The twisted glued square R^Λ no longer decomposes as a \mathbb{Z}^2 direct product. However, it decomposes as a direct sum $\bigoplus_{u \in \mathbb{Z}^2/\Lambda} R^\Lambda(u)$, where Λ is viewed as a lattice spanned by Λ_1, Λ_2 . The equivalence classes \mathbb{Z}^2/Λ correspond to the Spin^c structures over the surgery manifold on a link in S^3 .

4.2. Link surgery formula for a two-component link $\vec{L} = \vec{L}_1 \cup \vec{L}_2$. In order to denote the orientations of the sublinks, we use \pm signs to denote the positive and negative orientations, where the positive orientation is the induced orientation from \vec{L} and the negative orientation is the opposite orientation. Let $L = +L_1 \cup +L_2$, that is, L has the same orientation as \vec{L} .

The link surgery formula is the total complex of a square of chain complexes: the chain complexes at the vertices are the generalized Floer complexes described in Section 2.2, and the maps in the square are defined by means of *complete systems of hyperboxes* (see section 4.3 for the definition). From a complete system of hyperboxes of L , we get four sets of squares of chain complexes $R_{s,i,j}$, where $s \in \mathbb{H}(L)$, $i, j \in \{0, 1\}$:

$$\begin{aligned}
 (4.2) \quad R_{s,0,0} : \quad & \begin{array}{ccc} \mathfrak{A}^-(\mathcal{H}^L, s) & \xrightarrow{\Phi_s^{+L_1}} & \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{+L_1}(s)) \\ \Phi_s^{+L_2} \downarrow & \searrow \Phi_s^{+L_1 \cup +L_2} & \downarrow \Phi_{\psi^{+L_1}(s)}^{+L_2} \\ \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{+L_2}(s)) & \xrightarrow{\Phi_{\psi^{+L_2}(s)}^{+L_1}} & \mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{+L_1 \cup +L_2}(s)); \end{array}
 \end{aligned}$$

$$\begin{aligned}
 (4.3) \quad R_{s,1,0} : \quad & \begin{array}{ccc} \mathfrak{A}^-(\mathcal{H}^L, s) & \xrightarrow{\Phi_s^{-L_1}} & \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{-L_1}(s)) \\ \Phi_s^{+L_2} \downarrow & \searrow \Phi_s^{-L_1 \cup +L_2} & \downarrow \Phi_{\psi^{-L_1}(s)}^{+L_2} \\ \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{+L_2}(s)) & \xrightarrow{\Phi_{\psi^{+L_2}(s)}^{-L_1}} & \mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{-L_1 \cup +L_2}(s)); \end{array}
 \end{aligned}$$

$$\begin{aligned}
 (4.4) \quad R_{s,0,1} : \quad & \begin{array}{ccc} \mathfrak{A}^-(\mathcal{H}^L, s) & \xrightarrow{\Phi_s^{+L_1}} & \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{+L_1}(s)) \\ \Phi_s^{-L_2} \downarrow & \searrow \Phi_s^{+L_1 \cup -L_2} & \downarrow \Phi_{\psi^{+L_1}(s)}^{-L_2} \\ \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{-L_2}(s)) & \xrightarrow{\Phi_{\psi^{-L_2}(s)}^{+L_1}} & \mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{+L_1 \cup -L_2}(s)); \end{array}
 \end{aligned}$$

$$(4.5) \quad R_{s,1,1} : \quad \begin{array}{ccc} \mathfrak{A}^-(\mathcal{H}^L, s) & \xrightarrow{\Phi_s^{-L_1}} & \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{-L_1}(s)) \\ \Phi_s^{-L_2} \downarrow & \searrow \Phi_s^{-L_1 \cup -L_2} & \downarrow \Phi_{\psi^{-L_1}(s)}^{-L_2} \\ \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{-L_2}(s)) & \xrightarrow{\Phi_{\psi^{-L_2}(s)}^{-L_1}} & \mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{-L_1 \cup -L_2}(s)), \end{array}$$

where $\psi^{\vec{M}}$ is defined in Equation (2.1). Thus, we have four framed product squares with the natural framings

$$R_{i,j} = \prod_{s \in \mathbb{H}(L)} R_{s,i,j}.$$

For any surgery framing matrix Λ , the shifted squares $\{R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]\}_{i,j}$ are framed gluable according to Equations (4.1). The link surgery formula for the framed link (L, Λ) is the total complex of the Λ -twisted gluing of $\{R_{i,j}\}_{i,j}$ as follows,

$$(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-) := \begin{array}{ccc} \prod_{s \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^L, s) & \xrightarrow{\Phi^{+L_1} + \Phi^{-L_1}} & \prod_{s \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{+L_1}(s)) \\ \Phi^{+L_2} + \Phi^{-L_2} \downarrow & \begin{array}{c} \xrightarrow{\Phi^{+L_1 \cup -L_2} + \Phi^{-L_1 \cup -L_2}} \\ + \Phi^{+L_1 \cup +L_2} + \Phi^{-L_1 \cup +L_2} \end{array} & \downarrow \Phi^{+L_2} + \Phi^{-L_2} \\ \prod_{s \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{+L_2}(s)) & \xrightarrow{\Phi^{+L_1} + \Phi^{-L_1}} & \prod_{s \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{+L_1 \cup +L_2}(s)), \end{array}$$

where $\Phi^\circ = \prod_{s \in \mathbb{H}(L)} \Phi_s^\circ$, $\circ = \pm L_1, \pm L_2, \pm L_1 \cup \pm L_2$.

The map

$$\Phi_s^{\vec{M}} : \mathfrak{A}^-(\mathcal{H}^L, s) \rightarrow \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^{\vec{M}}(s))$$

is defined by

$$(4.6) \quad \Phi_s^{\vec{M}} = D_{p^{\vec{M}}(s)}^{\vec{M}} \circ \mathcal{I}_s^{\vec{M}}.$$

We will spell out the constructions of $D_{p^{\vec{M}}(s)}^{\vec{M}}$ and $\mathcal{I}_s^{\vec{M}}$ in the next sections by using *primitive systems of hyperboxes*.

For the Λ -twisted gluing of squares, there is a direct sum splitting of the complex

$$\mathcal{C}^-(\mathcal{H}, \Lambda) = \bigoplus_{\mathfrak{u} \in \mathbb{H}(L)/H(L, \Lambda)} \mathcal{C}^-(\mathcal{H}, \Lambda, \mathfrak{u}),$$

where we identify $\mathbb{H}(L)/H(L, \Lambda)$ with $\text{Spin}^c(S_\Lambda^3(L))$.

Theorem 4.15 (Manolescu-Ozsváth Link Surgery Theorem, Theorem 7.7 of [5] for two-component links). *Fix a primitive system of hyperboxes \mathcal{H} for an oriented two-component link \vec{L} in S^3 , and fix a framing Λ of L . Then for any $\mathfrak{u} \in \text{Spin}^c(S_\Lambda^3(L)) \cong \mathbb{H}(L)/H(L, \Lambda)$, there is an isomorphism of $\mathbb{F}[[U]]$ -modules*

$$H_*(\mathcal{C}^-(\mathcal{H}, \Lambda, \mathfrak{u}), \mathcal{D}^-) \cong \mathbf{HF}_*(S_\Lambda^3(L), \mathfrak{u}),$$

where $\mathbb{F}[[U]] = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$.

Here, we let \mathbf{HF}^- denote the completion of HF^- with respect to the maximal ideal (U) in the ring $\mathbb{F}[[U]]$. Since completion is an exact functor, \mathbf{HF}^- can be regarded as the homology of the

complex $\mathbf{CF}^- = CF^- \otimes_{\mathbb{F}[U]} \mathbb{F}[[U]]$, where $\mathbb{F}[[U]]$ is the completion of $\mathbb{F}[U]$. When s is a torsion Spin^c structure of a 3-manifold M , if

$$\mathbf{HF}^-(M, s) = \mathbb{F}[[U]] \oplus T$$

with T a torsion $\mathbb{F}[[U]]$ -module, then

$$HF^-(M, s) = \mathbb{F}[U] \oplus T.$$

For more details, see Section 2 in [5].

Remark 4.16. The link surgery theorem states that all the U_i -actions are the same in the homology of the surgery complex.

4.3. Inclusion maps and destabilization maps.

4.3.1. *Inclusion maps.* In the link surgery formula, we need a set of chain maps $\mathcal{I}_s^{\vec{M}}$ in (4.6) which are called *inclusion maps*. Here, we define the inclusion maps for all links with arbitrary number of components. In the knot case, the inclusion maps correspond to the maps h_s and v_s from [15].

Definition 4.17. Let \vec{M} be an oriented sublink of \vec{L} . Define

$$\begin{aligned} I_+(\vec{L}, \vec{M}) &= \{i : \vec{L} \text{ and } \vec{M} \text{ share the same orientation on } L_i\}; \\ I_-(\vec{L}, \vec{M}) &= \{i : \vec{L} \text{ and } \vec{M} \text{ have different orientations on } L_i\}. \end{aligned}$$

A projection map $p^{\vec{M}} : \mathbb{H}(L) \rightarrow \mathbb{H}(L)$ is defined component-wisely as follows:

$$p_i^{\vec{M}}(s) = \begin{cases} +\infty & \text{if } i \in I_+(\vec{L}, \vec{M}) \\ -\infty & \text{if } i \in I_-(\vec{L}, \vec{M}) \\ s & \text{otherwise} \end{cases}.$$

Definition 4.18 (Inclusion maps). Suppose $\vec{M} \subset \vec{L}$ is an oriented sublink, and $s = (s_1, s_2) \in \mathbb{H}(L)$ satisfies $s_i \neq \mp\infty$ for those $i \in I_{\pm}(\vec{L}, \vec{M})$. Let \mathcal{H} be a Heegaard diagram of L . The *inclusion map* $\mathcal{I}_s^{\vec{M}} : \mathfrak{A}^-(\mathcal{H}^L, s) \rightarrow \mathfrak{A}^-(\mathcal{H}^L, p^{\vec{M}}(s))$ is defined by the formula

$$\mathcal{I}_s^{\vec{M}}(x) = \prod_{i \in I_+(\vec{L}, \vec{M})} U_{\tau_i}^{\max(A_i(x) - s_i, 0)} \prod_{i \in I_-(\vec{L}, \vec{M})} U_{\tau_i}^{\max(s_i - A_i(x), 0)} x.$$

One can verify this is a chain map.

Example 4.19. Suppose \vec{L} is a two-component link and \mathcal{H} is a basic Heegaard diagram of L . We have the following inclusion maps

- $\mathcal{I}_{s_1, s_2}^{+L_1} : A_{s_1, s_2}^- \rightarrow A_{+\infty, s_2}^-$, $\mathcal{I}_{s_1, s_2}^{-L_1} : A_{s_1, s_2}^- \rightarrow A_{-\infty, s_2}^-$.
- $\mathcal{I}_{s_1, s_2}^{+L_2} : A_{s_1, s_2}^- \rightarrow A_{s_1, +\infty}^-$, $\mathcal{I}_{s_1, s_2}^{-L_2} : A_{s_1, s_2}^- \rightarrow A_{s_1, -\infty}^-$.
- $\mathcal{I}_{s_1, s_2}^{+L_1 \cup L_2} : A_{s_1, s_2}^- \rightarrow A_{+\infty, +\infty}^-$, $\mathcal{I}_{s_1, s_2}^{-L_1 \cup L_2} : A_{s_1, s_2}^- \rightarrow A_{-\infty, +\infty}^-$,
 $\mathcal{I}_{s_1, s_2}^{+L_1 \cup -L_2} : A_{s_1, s_2}^- \rightarrow A_{+\infty, -\infty}^-$, $\mathcal{I}_{s_1, s_2}^{-L_1 \cup -L_2} : A_{s_1, s_2}^- \rightarrow A_{-\infty, -\infty}^-$.

4.3.2. *Destabilization maps.* Let $\vec{L} = \vec{L}_1 \cup \vec{L}_2$. We set

$$J(+L_i) = \{(s_1, s_2) \in \mathbb{H}(L) | s_i = +\infty\}, \quad J(-L_i) = \{(s_1, s_2) \in \mathbb{H}(L) | s_i = -\infty\}.$$

Now let $\vec{M} \subset L$ be an oriented sublink, and let $\{\vec{M}_i\}_i$ be all the oriented components of \vec{M} . Define

$$J(\vec{M}) = \bigcap_i J(\vec{M}_i).$$

For $s \in J(\vec{M})$, there is a *destabilization map*

$$D_s^{\vec{M}} : \mathfrak{A}^-(\mathcal{H}^L, s) \rightarrow \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^{\vec{M}}(s)),$$

which gives rise to the map $D_{p^{\vec{M}}(s)}^{\vec{M}}$ in (4.6). Note that $p^{\vec{M}}(s) \in J(\vec{M})$ for any $s \in \mathbb{H}(L)$. In the knot case, the destabilization map corresponds to the map identifying $C\{i > 0\}$ and $C\{j > 0\}$. We will give the definition in the next section.

Example 4.20. Let $s = (s_1, s_2) \in \mathbb{H}(L)$ and $\vec{M} = \pm L_1$, then $p^{\pm L_1}(s) = (\pm\infty, s_2)$. The destabilization map

$$D_{\pm\infty, s_2}^{\pm L_1} : \mathfrak{A}^-(\mathcal{H}^L, \pm\infty, s_2) \rightarrow \mathfrak{A}^-(\mathcal{H}^{L_2}, s_2 - \frac{\text{lk}(+L_1, \pm L_2)}{2})$$

is a chain homotopy equivalence.

If we consider sublinks $\vec{M} = \pm L_1 \cup \pm L_2$, then we will get destabilization maps from $\mathfrak{A}^-(\mathcal{H}^L, \pm\infty, \pm\infty)$ to $\mathfrak{A}^-(\mathcal{H}^\emptyset, 0)$, namely,

$$\begin{aligned} D_{+\infty, +\infty}^{+L_1 \cup +L_2} &: \mathfrak{A}^-(\mathcal{H}^L, +\infty, +\infty) \rightarrow \mathfrak{A}^-(\mathcal{H}^\emptyset, 0), \\ D_{-\infty, +\infty}^{-L_1 \cup +L_2} &: \mathfrak{A}^-(\mathcal{H}^L, -\infty, +\infty) \rightarrow \mathfrak{A}^-(\mathcal{H}^\emptyset, 0), \\ D_{+\infty, -\infty}^{+L_1 \cup -L_2} &: \mathfrak{A}^-(\mathcal{H}^L, +\infty, -\infty) \rightarrow \mathfrak{A}^-(\mathcal{H}^\emptyset, 0), \\ D_{-\infty, -\infty}^{-L_1 \cup -L_2} &: \mathfrak{A}^-(\mathcal{H}^L, -\infty, -\infty) \rightarrow \mathfrak{A}^-(\mathcal{H}^\emptyset, 0). \end{aligned}$$

4.3.3. *Primitive system of hyperboxes.* In [5], *complete system of hyperboxes* is defined in order to define the destabilization maps.

Definition 4.21 (Complete pre-system of hyperboxes). A *complete pre-system of hyperboxes* \mathcal{H} representing the link \vec{L} consists of a collection of hyperboxes of Heegaard diagrams, subject to certain compatibility conditions as follows. For each pair of subsets $M \subseteq L' \subseteq L$, and each orientation \vec{M} on M , the complete pre-system assigns a hyperbox $\mathcal{H}^{\vec{L}', \vec{M}}$ for the pair (\vec{L}', \vec{M}) , where \vec{L}' has the induced orientation from \vec{L} . Moreover, the hyperbox $\mathcal{H}^{\vec{L}', \vec{M}}$ is required to be compatible with both $\mathcal{H}^{\vec{L}, \vec{M}'}$ and $\mathcal{H}^{\vec{L}' - \vec{M}', \vec{M} - \vec{M}'}$.

When the sublink \vec{L}' has the induced orientation from \vec{L} , we simply denote it by L' . Thus, we use notation $\mathcal{H}^{L', \vec{M}} = \mathcal{H}^{\vec{L}', \vec{M}}$.

In the above definition, there is some compatibility condition we have not spelled out. A *complete system of hyperboxes* is a complete pre-system with some additional conditions regarding the surface isotopies connecting those hyperboxes. We will not explain them here. Instead, we give a special complete system of hyperboxes for two-component links satisfying these conditions, which illustrates the main idea. They are called *primitive system of hyperboxes*.

In a complete pre-system \mathcal{H} , we have four zero dimensional hyperboxes of Heegaard diagrams, $\mathcal{H}^{L, \emptyset}, \mathcal{H}^{L_1, \emptyset}, \mathcal{H}^{L_2, \emptyset}, \mathcal{H}^{\emptyset, \emptyset}$, where $\mathcal{H}^{L, \emptyset}$ is a Heegaard diagram of L , $\mathcal{H}^{L_i, \emptyset}$ is a Heegaard diagram of L_i , and $\mathcal{H}^{\emptyset, \emptyset}$ is a Heegaard diagram of S^3 . We denote the four Heegaard diagrams simply by $\mathcal{H}^L, \mathcal{H}^{L_1}, \mathcal{H}^{L_2}, \mathcal{H}^\emptyset$.

In a primitive system, $\mathcal{H}^L = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_1, z_2\})$ is a basic Heegaard diagram for L , \mathcal{H}^{L_i} is obtained from \mathcal{H}^L by deleting z_i , and \mathcal{H}^\emptyset is obtained from \mathcal{H}^L by deleting z_1, z_2 . A primitive system of hyperboxes also has two non-trivial 1-dimensional hyperboxes of strongly equivalent Heegaard diagrams of size (2) and one non-trivial 2-dimensional hyperboxes of strongly equivalent Heegaard diagrams of size (2, 2). They induce the corresponding hyperboxes of Floer chain complexes $\prod_s \mathfrak{A}^-(\mathcal{H}^{L'}, s)$.

Given a basic Heegaard diagram $\mathcal{H}^L = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_1, z_2\})$ of $\vec{L} = \vec{L}_1 \cup \vec{L}_2$, from Equation (2.2), we see $\mathfrak{A}^-(\mathcal{H}^L, (+\infty, s_2))$ is counting $n_{w_1}(\phi)$ without using z_1 , thus as the same as deleting

z_1 . Moreover $\mathcal{H}^{L_2} = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_2\})$ is a Heegaard diagram of \vec{L}_2 with one free basepoint w_1 . We call this diagram the *reduction* of \mathcal{H}^L at $+L_1$, denoted by $r_{+L_1}(\mathcal{H}^L)$; see [5] Definition 4.17. Hence, we have an identification between $\mathfrak{A}^-(\mathcal{H}^L, (+\infty, s_2))$ and $\mathfrak{A}^-(r_{+L_1}(\mathcal{H}^L), s_2 - \frac{\text{lk}(+L_1, +L_2)}{2})$. Similarly, we define $r_{-L_1}(\mathcal{H}^L)$ to be the diagram obtained from \mathcal{H}^L by deleting w_1 and relabeling z_1 as w_1 . We have an identification between $\mathfrak{A}^-(\mathcal{H}^L, (-\infty, s_2))$ and $\mathfrak{A}^-(r_{-L_1}(\mathcal{H}^L), s_2 - \frac{\text{lk}(-L_1, +L_2)}{2})$, since $\mathfrak{A}^-(\mathcal{H}^L, (-\infty, s_2))$ uses basepoints $\{z_1, w_2\} \subset \mathcal{H}^L$.

Moreover, the diagrams $r_{-L_1}(\mathcal{H}^L)$ and $r_{+L_1}(\mathcal{H}^L)$ are related by Heegaard moves, for they represent the same knot L_2 . By definition, there is an arc c in $\Sigma - \alpha$ connecting w_1 and z_1 , so we can move z_1 along c to w_1 , by a sequence of Heegaard moves. Moving a basepoint to cross some β -curve can be done by a sequence of handleslides and isotopies of β -curves, stabilizations, and destabilization followed by a surface isotopy. However, if we need stabilizations/destabilizations, we can modify the original Heegaard diagram \mathcal{H}^L by these stabilizations in the beginning. Thus, we can always get a diagram $\tilde{\mathcal{H}}^L$, such that there is a sequence of Heegaard moves only of handleslides and isotopies of β -curves together with some surface isotopy from $r_{-L_1}(\tilde{\mathcal{H}}^L)$ to $r_{+L_1}(\tilde{\mathcal{H}}^L)$. In sum, there is an surface isotopy $h : \Sigma \rightarrow \Sigma$ supported in a small neighborhood of c (so h fixes other basepoints and all the α -curves), such that $h(w_1) = z_1$ and $h(r_{+L_1}(\tilde{\mathcal{H}}^L))$ is strongly equivalent to $r_{-L_1}(\tilde{\mathcal{H}}^L)$ via handleslides and isotopies of β -curves.

Definition 4.22 (Primitive Heegaard diagrams). For any basic Heegaard diagram \mathcal{H} of an oriented link $\vec{L} = \vec{L}_1 \cup \vec{L}_2$, there are surface isotopies $h_i^{\mathcal{H}} : \Sigma \rightarrow \Sigma$ supported in a small neighborhood of the arc c_i connecting w_i and z_i in $\Sigma - \alpha$, such that $h_i^{\mathcal{H}}(w_i) = z_i$ and $h_i^{\mathcal{H}}$ preserves the α -curves and the other basepoints. They are unique up to isotopy. The basic Heegaard diagram \mathcal{H} is called *primitive*, if it is admissible and $r_{-L_i}(\mathcal{H})$ is strongly equivalent to $h_i^{\mathcal{H}}(r_{+L_i}(\mathcal{H}))$ for both $i = 1, 2$.

From the above discussion, we can get the following lemma.

Lemma 4.23. *Let L be an oriented two-component link, and let \mathcal{H} be a basic admissible Heegaard diagram of L . Then there is an index one/two stabilization turning \mathcal{H} into a primitive Heegaard diagram $\tilde{\mathcal{H}}$.*

Fixing a primitive Heegaard diagram \mathcal{H}^L for $\vec{L} = \vec{L}_1 \cup \vec{L}_2$, we can get two sequences of strongly equivalent Heegaard diagrams $\bar{\mathcal{H}}^{L, -L_i}$:

$$(4.7) \quad \bar{\mathcal{H}}^{L, -L_1} : r_{-L_1}(\mathcal{H}^L) = \bar{\mathcal{H}}^{L, -L_1}(\emptyset) \rightarrow \bar{\mathcal{H}}^{L, -L_1}(L_1) = h_1^{\mathcal{H}}(r_{+L_1}(\mathcal{H}^L)),$$

$$(4.8) \quad \bar{\mathcal{H}}^{L, -L_2} : r_{-L_2}(\mathcal{H}^L) = \bar{\mathcal{H}}^{L, -L_2}(\emptyset) \rightarrow \bar{\mathcal{H}}^{L, -L_2}(L_2) = h_2^{\mathcal{H}}(r_{+L_2}(\mathcal{H}^L)).$$

These induce another two sequences of strongly equivalent Heegaard diagrams $\bar{\mathcal{H}}^{L_i, -L_i}$:

$$(4.9) \quad \bar{\mathcal{H}}^{L_1, -L_1} : r_{-L_1}(r_{+L_2}(\mathcal{H}^L)) = \bar{\mathcal{H}}^{L_1, -L_1}(\emptyset) \rightarrow \bar{\mathcal{H}}^{L_1, -L_1}(L_1) = h_1^{\mathcal{H}}(r_{+L_1}(r_{+L_2}(\mathcal{H}^L))),$$

$$(4.10) \quad \bar{\mathcal{H}}^{L_2, -L_2} : r_{-L_2}(r_{+L_1}(\mathcal{H}^L)) = \bar{\mathcal{H}}^{L_2, -L_2}(\emptyset) \rightarrow \bar{\mathcal{H}}^{L_2, -L_2}(L_2) = h_2^{\mathcal{H}}(r_{+L_2}(r_{+L_1}(\mathcal{H}^L))),$$

together with a square of strongly equivalent Heegaard diagrams $\bar{\mathcal{H}}^{L, -L_1 \cup -L_2}$:

$$(4.11) \quad \begin{array}{ccc} r_{-L_1 \cup -L_2}(\mathcal{H}) = \bar{\mathcal{H}}^{L_1 \cup L_2, -L_1 \cup -L_2}(\emptyset) & \xrightarrow{\quad} & \bar{\mathcal{H}}^{L_1 \cup L_2, +L_1 \cup -L_2}(L_1) = h_1^{\mathcal{H}}(r_{+L_1 \cup -L_2}(\mathcal{H})) \\ \downarrow & \searrow & \downarrow \\ h_2^{\mathcal{H}}(r_{-L_1 \cup +L_2}(\mathcal{H})) = \bar{\mathcal{H}}^{L_1 \cup L_2, -L_1 \cup -L_2}(L_2) & \xrightarrow{\quad} & \bar{\mathcal{H}}^{L_1 \cup L_2, -L_1 \cup -L_2}(L) = h_1^{\mathcal{H}} \circ h_2^{\mathcal{H}}(r_{+L_1 \cup +L_2}(\mathcal{H})). \end{array}$$

These almost produce a complete system of hyperbox $\bar{\mathcal{H}}$ except for the admissibility of $\bar{\mathcal{H}}$. We call this system a *primitive almost complete system of hyperbox*.

Definition 4.24 (Primitive almost complete system of hyperbox). Given a primitive Heegaard diagram $\mathcal{H}^L = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_1, z_2\})$ of $\vec{L} = \vec{L}_1 \cup \vec{L}_2$, there exists a *primitive almost complete system of hyperbox* \mathcal{H} associated to \mathcal{H}^L consisting of

- four 0-dimensional hyperboxes of Heegaard diagrams:

$$\bar{\mathcal{H}}^L = \mathcal{H}, \quad \bar{\mathcal{H}}^{L_1} = r_{+L_2}(\mathcal{H}), \quad \bar{\mathcal{H}}^{L_2} = r_{+L_1}(\mathcal{H}), \quad \bar{\mathcal{H}}^\emptyset = r_{+L_1 \cup +L_2}(\mathcal{H});$$

- eight 1-dimensional hyperboxes of Heegaard diagrams:

$$\bar{\mathcal{H}}^{L, \pm L_i}, \quad \bar{\mathcal{H}}^{L_i, \pm L_i}, \quad \forall i = 1, 2,$$

where $\bar{\mathcal{H}}^{L, +L_i}, \bar{\mathcal{H}}^{L_i, +L_i}$ are trivial hyperboxes, i.e. just a Heegaard diagram, and $\bar{\mathcal{H}}^{L, -L_i}, \bar{\mathcal{H}}^{L_i, -L_i}$ are described in Equations (4.7) and (4.9);

- four 2-dimensional hyperboxes of Heegaard diagrams: one trivial hyperbox $\bar{\mathcal{H}}^{L, +L_1 \cup +L_2}$, two degenerate hyperboxes:

$$\bar{\mathcal{H}}^{L, +L_1 \cup -L_2} = \bar{\mathcal{H}}^{L_2, -L_2}, \quad \bar{\mathcal{H}}^{L, -L_1 \cup +L_2} = \bar{\mathcal{H}}^{L_1, -L_1},$$

and a square of strongly equivalent Heegaard diagrams $\mathcal{H}^{L, -L_1 \cup -L_2}$, which is described in Equation (4.11).

Definition 4.25 (Primitive system of hyperboxes). Given a primitive diagram \mathcal{H}^L and the induced primitive almost complete systems of hyperboxes \mathcal{H} , if the admissibility of \mathcal{H} is not satisfied, we can enlarge the hyperbox in \mathcal{H} to achieve admissibility, thus getting a complete system of hyperboxes. We call the result a *primitive system of hyperboxes* \mathcal{H} .

Indeed, if $\bar{\mathcal{H}}^{L, -L_1}$ is not admissible, i.e. $(\Sigma, \alpha, \beta, \beta', \mathbf{w}, \mathbf{z})$ is not admissible, then we can insert an isotopy of β, β'' , such that both $(\Sigma, \alpha, \beta, \beta'', \mathbf{w}, \mathbf{z})$ and $(\Sigma, \alpha, \beta'', \beta', \mathbf{w}, \mathbf{z})$ are admissible. Suppose $\{D_1, \dots, D_m\}$ is a basis of the \mathbb{Q} -vector space of the periodic domains in $(\Sigma, \alpha, \beta, \beta', \mathbf{w}, \mathbf{z})$ with only positive multiplicities. Let D_1^c be the union of all the regions which are not in D_1 . Then $D_1^c \neq \emptyset$, since $n_{\mathbf{w}} D_1 = 0$. As $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ and $(\Sigma, \alpha, \beta', \mathbf{w}, \mathbf{z})$ are both admissible, the boundary of D_1 must contain a β -curve and a β' -curve. Thus there is a β -arc b and a β' -arc b' on $D_1 \cap D_1^c$. So we can find a path γ in D_1 connecting b to b' without touching the β -curves, and then do a finger move of the β -curve containing b along γ to get negative multiplicities for D_1 (see [17] for the definition of finger move). Similarly we deal with the other D_i 's. Finally, the new β in the above process is chosen to be β'' . Similar arguments work for the case of the square $\mathcal{H}^{L, -L_1 \cup -L_2}$.

Therefore to achieve admissibility, we can enlarge the square of Heegaard diagram $\bar{\mathcal{H}}^{L, -L_1 \cup -L_2}$:

$$\begin{array}{ccc} (\Sigma, \alpha, \beta_{11}, \mathbf{w}, \mathbf{z}) & \rightarrow & (\Sigma, \alpha, \beta_{13}, \mathbf{w}, \mathbf{z}) \\ \downarrow & \searrow & \downarrow \\ (\Sigma, \alpha, \beta_{31}, \mathbf{w}, \mathbf{z}) & \rightarrow & (\Sigma, \alpha, \beta_{33}, \mathbf{w}, \mathbf{z}) \end{array}$$

into $\mathcal{H}^{L, -L_1 \cup -L_2}$:

$$\begin{array}{ccccc} (\Sigma, \alpha, \beta_{11}, \mathbf{w}, \mathbf{z}) & \rightarrow & (\Sigma, \alpha, \beta_{12}, \mathbf{w}, \mathbf{z}) & \rightarrow & (\Sigma, \alpha, \beta_{13}, \mathbf{w}, \mathbf{z}) \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ (\Sigma, \alpha, \beta_{21}, \mathbf{w}, \mathbf{z}) & \rightarrow & (\Sigma, \alpha, \beta_{22}, \mathbf{w}, \mathbf{z}) & \rightarrow & (\Sigma, \alpha, \beta_{23}, \mathbf{w}, \mathbf{z}) \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ (\Sigma, \alpha, \beta_{31}, \mathbf{w}, \mathbf{z}) & \rightarrow & (\Sigma, \alpha, \beta_{32}, \mathbf{w}, \mathbf{z}) & \rightarrow & (\Sigma, \alpha, \beta_{33}, \mathbf{w}, \mathbf{z}) \end{array}$$

In order to send every hyperbox of Heegaard diagrams $\mathcal{H}^{\vec{L}, \vec{M}}$ to a hyperbox of chain complexes $\mathfrak{A}^-(\mathcal{H}^{\vec{L}, \vec{M}}, \mathbf{s})$, we need a set of Θ chain elements. We call the choice of these Θ -elements a *filling* of the hyperboxes Heegaard diagrams. Let us explain Θ -elements case by case.

For $\mathcal{H}^{L, -L_1}$, we have a sequence of strongly equivalent Heegaard diagrams of $\vec{L} - L_1$,

$$\mathcal{H}^{L, -L_1} : (\Sigma, \alpha, \beta_1, \mathbf{w}, \mathbf{z}) \rightarrow (\Sigma, \alpha, \beta_2, \mathbf{w}, \mathbf{z}) \rightarrow (\Sigma, \alpha, \beta_3, \mathbf{w}, \mathbf{z}).$$

We choose a cycle element $\Theta_{\beta_1, \beta_2}$ representing the maximal degree element in the homology of $\mathfrak{A}^-(\mathbb{T}_{\beta_1}, \mathbb{T}_{\beta_2}, 0)$. Then we define a chain homotopy equivalence $D_{\beta_1, \beta_2} : \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta_1}, s) \rightarrow \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta_2}, s)$ by using triangle maps $D_{\beta_1, \beta_2}(\mathbf{x}) = f_{\alpha\beta_1\beta_2}(\mathbf{x} \otimes \Theta_{\beta_1, \beta_2})$. Similarly, we get a chain homotopy equivalence $D_{\beta_2, \beta_3} : \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta_2}, s) \rightarrow \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta_3}, s)$ by choosing $\Theta_{\beta_2, \beta_3} \in \mathfrak{A}^-(\mathbb{T}_{\beta_2}, \mathbb{T}_{\beta_3}, 0)$. Thus, $D^{-L_1} = D_{\beta_2, \beta_3} \circ D_{\beta_1, \beta_2}$ is also a chain homotopy equivalence. Let us put a subscript on D^{-L_1} for labeling the Spin^c structure. Since $\mathfrak{A}^-(\mathcal{H}^L, (+\infty, s_2)) = \mathfrak{A}(r_{+L_1}(\mathcal{H}^L), s_2 - \frac{\text{lk}(+L_1, +L_2)}{2})$, $\mathfrak{A}^-(\mathcal{H}^L, (-\infty, s_2)) = \mathfrak{A}^-(r_{-L_1}(\mathcal{H}^L), s_2 - \frac{\text{lk}(-L_1, +L_2)}{2})$, we write

$$D_{-\infty, s_2}^{-L_1} : \mathfrak{A}^-(\mathcal{H}^L, (-\infty, s_2)) \rightarrow \mathfrak{A}^-(\mathcal{H}^L, (+\infty, s_2 + \text{lk}(+L_1, +L_2))),$$

or simply

$$D_{-\infty, s_2}^{-L_1} : A_{-\infty, s_2}^- \rightarrow A_{+\infty, s_2 + \text{lk}}^-.$$

Similarly we define $D_{s_1, -\infty}^{-L_2} : A_{s_1, -\infty}^- \rightarrow A_{s_1 + \text{lk}, +\infty}^-$.

For the 2-dimensional hyperbox of Heegaard diagrams $\mathcal{H}^{L, -L_1 \cup -L_2}$, we can get a square of chain complexes. Let us first look at the upper left quarter of $\mathcal{H}^{L, -L_1 \cup -L_2}$:

$$\begin{array}{ccc} (\Sigma, \alpha, \beta_{11}, \mathbf{w}, \mathbf{z}) & \xrightarrow{\Theta_{\beta_{11}\beta_{12}}} & (\Sigma, \alpha, \beta_{12}, \mathbf{w}, \mathbf{z}) \\ \Theta_{\beta_{11}\beta_{21}} \downarrow & \searrow \Theta_{\beta_{11}\beta_{22}} & \downarrow \Theta_{\beta_{12}\beta_{22}} \\ (\Sigma, \alpha, \beta_{21}, \mathbf{w}, \mathbf{z}) & \xrightarrow{\Theta_{\beta_{21}\beta_{22}}} & (\Sigma, \alpha, \beta_{22}, \mathbf{w}, \mathbf{z}). \end{array}$$

In the above the diagram, the Θ -elements on the edges are arbitrary cycles representing the maximal degree elements in their homology groups. Let $c = f_{\beta_{11}\beta_{12}\beta_{22}}(\Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}})$. The equation

$$\begin{aligned} & \partial(f_{\beta_{11}\beta_{12}\beta_{22}}(\Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}})) = \\ & f_{\beta_{11}\beta_{12}\beta_{22}}((\partial\Theta_{\beta_{11}\beta_{12}}) \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes (\partial\Theta_{\beta_{21}\beta_{22}})) = 0 \end{aligned}$$

shows that c is a cycle in $\mathfrak{A}_{\mu}^-(\mathbb{T}_{\beta_{11}}, \mathbb{T}_{\beta_{22}}, 0)$, where μ equals to the maximal degree of the homology of $\mathfrak{A}^-(\mathbb{T}_{\beta_{11}}, \mathbb{T}_{\beta_{22}}, 0)$. Since the curves β_{**} are all strongly equivalent, up to chain homotopy equivalences, we can only consider the case when they are all small Hamiltonian isotopies of each other. By Lemma 9.7 in [12], we can see that $f_{\beta_{11}\beta_{12}\beta_{22}}(\Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}), f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}})$ both represent the maximal degree element in the homology of $CF(\mathbb{T}_{\beta_{11}}, \mathbb{T}_{\beta_{22}})$. Thus, c is 0 in the homology. So there is a $\Theta_{\beta_{11}, \beta_{22}}$ such that $\partial\Theta_{\beta_{11}, \beta_{22}} = c$, where $\partial = f_{\beta_{11}, \beta_{22}}$. In sum,

$$f_{\beta_{11}\beta_{12}\beta_{22}}(\Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}}) = f_{\beta_{11}\beta_{22}}(\Theta_{\beta_{11}\beta_{22}}).$$

From the quadratic A_{∞} associativity Equation (2.3), we have a square of chain complexes

$$\begin{array}{ccc} \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta_{11}}, s) & \xrightarrow{D_{\beta_{11}\beta_{12}}} & \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta_{12}}, s) \\ D_{\beta_{11}\beta_{21}} \downarrow & \searrow D_{\beta_{11}\beta_{22}} & \downarrow D_{\beta_{12}\beta_{22}} \\ \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta_{21}}, s) & \xrightarrow{D_{\beta_{21}\beta_{22}}} & \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta_{22}}, s), \end{array}$$

where $D_{\beta_{11}\beta_{22}}(x) = f_{\alpha\beta_{11}\beta_{12}\beta_{22}}(x \otimes \Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\alpha\beta_{11}\beta_{21}\beta_{22}}(x \otimes \Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}}) + f_{\alpha\beta_{11}\beta_{22}}(x \otimes \Theta_{\beta_{11}\beta_{22}})$, and

$$\begin{aligned} D_{\beta_{11}\beta_{12}}(x) &= f_{\alpha\beta_{11}\beta_{12}}(x \otimes \Theta_{\beta_{11}\beta_{12}}), \quad D_{\beta_{12}\beta_{22}}(x) = f_{\alpha\beta_{12}\beta_{22}}(x \otimes \Theta_{\beta_{12}\beta_{22}}), \\ D_{\beta_{11}\beta_{21}}(x) &= f_{\alpha\beta_{11}\beta_{21}}(x \otimes \Theta_{\beta_{11}\beta_{21}}), \quad D_{\beta_{11}\beta_{22}}(x) = f_{\alpha\beta_{11}\beta_{22}}(x \otimes \Theta_{\beta_{11}\beta_{22}}). \end{aligned}$$

Similarly, we can choose other Θ -elements on $\mathcal{H}^{L, -L_1 \cup -L_2}$, and get a rectangle of chain complexes of size $(2, 2)$. We denote it by $\mathfrak{A}^-(\mathcal{H}^{L, -L_1 \cup -L_2}, s)$.

Definition 4.26. We define the destabilization map $D^{-L_1 \cup -L_2}$ to be the diagonal map in the compression of $\mathfrak{A}^-(\mathcal{H}^{L, -L_1 \cup -L_2}, s)$:

$$\begin{array}{ccc} \mathfrak{A}^-(r_{-L_1 \cup -L_2}(\mathcal{H}^L), s) & \longrightarrow & \mathfrak{A}^-(r_{-L_1 \cup +L_2}(\mathcal{H}^L), s) \\ \downarrow & \searrow & \downarrow \\ \mathfrak{A}^-(r_{+L_1 \cup -L_2}(\mathcal{H}^L), s) & \longrightarrow & \mathfrak{A}^-(r_{+L_1 \cup +L_2}(\mathcal{H}^L), s). \end{array}$$

Since $\mathfrak{A}^-(r_{-L_1 \cup -L_2}(\mathcal{H}^L), s) = \mathfrak{A}^-(\mathcal{H}^L, -\infty, -\infty)$, we denote it by $D_{-\infty, -\infty}^{-L_1 \cup -L_2}$.

As all the other hyperboxes of Heegaard diagrams are trivial, the following identities hold

$$D_{+\infty, s_2}^{+L_1} = id, D_{s_1, +\infty}^{+L_2} = id, D_{-\infty, +\infty}^{-L_1 \cup +L_2} = 0, D_{+\infty, -\infty}^{+L_1 \cup -L_2} = 0, D_{+\infty, +\infty}^{+L_1 \cup +L_2} = 0.$$

Now we can build all the rectangles of chain complexes as follows, where $lk = lk(\vec{L}_1, \vec{L}_2)$.

(4.12)

$$\begin{array}{ccc} \mathfrak{R}_{s,1,1} := & & \mathfrak{R}_{s,0,1} := \\ \begin{array}{ccccc} A_{s_1, s_2}^- & \xrightarrow{I_{s_1, s_2}^{-L_2}} & A_{s_1, -\infty}^- & \xrightarrow{D_{s_1, -\infty}^{-L_2}} & A_{s_1 + lk, +\infty}^- \\ I_{s_1, s_2}^{-L_1} \downarrow & \searrow I_{s_1, s_2}^{-L_1 \cup -L_2} & \downarrow I_{s_1, -\infty}^{-L_1} & & \downarrow I_{s_1 + lk, +\infty}^{-L_1} \\ A_{-\infty, s_2}^- & \xrightarrow{I_{-\infty, s_2}^{-L_2}} & A_{-\infty, -\infty}^- & \xrightarrow{D_{-\infty, -\infty}^{-L_2}} & A_{-\infty, +\infty}^- \\ D_{-\infty, s_2}^{-L_1} \downarrow & \searrow D_{-\infty, -\infty}^{-L_1 \cup -L_2} & \downarrow D_{-\infty, -\infty}^{-L_1} & & \downarrow D_{-\infty, +\infty}^{-L_1} \\ A_{+\infty, s_2 + lk}^- & \xrightarrow{I_{+\infty, s_2 + lk}^{-L_2}} & A_{+\infty, -\infty}^- & \xrightarrow{D_{+\infty, -\infty}^{-L_2}} & A_{+\infty, +\infty}^- \end{array} & & \begin{array}{ccccc} A_{s_1, s_2}^- & \xrightarrow{I_{s_1, s_2}^{-L_2}} & A_{s_1, -\infty}^- & \xrightarrow{D_{s_1, -\infty}^{-L_2}} & A_{s_1 + lk, +\infty}^- \\ I_{s_1, s_2}^{+L_1} \downarrow & \searrow I_{s_1, s_2}^{+L_1 \cup -L_2} & \downarrow I_{s_1, -\infty}^{+L_1 \cup -L_2} & & \downarrow I_{s_1 + lk, +\infty}^{+L_1} \\ A_{+\infty, s_2}^- & \xrightarrow{I_{+\infty, s_2}^{-L_2}} & A_{+\infty, -\infty}^- & \xrightarrow{D_{+\infty, -\infty}^{-L_2}} & A_{+\infty, +\infty}^- \\ id \downarrow & \searrow id & \downarrow id & & \downarrow id \\ A_{+\infty, s_2}^- & \xrightarrow{I_{+\infty, s_2}^{-L_2}} & A_{+\infty, -\infty}^- & \xrightarrow{D_{+\infty, -\infty}^{-L_2}} & A_{+\infty, +\infty}^- \end{array} \end{array}$$

(4.13)

$$\begin{array}{ccc} \mathfrak{R}_{s,1,0} := & & \mathfrak{R}_{s,0,0} := \\ \begin{array}{ccccc} A_{s_1, s_2}^- & \xrightarrow{I_{s_1, s_2}^{+L_2}} & A_{s_1, +\infty}^- & \xrightarrow{id} & A_{s_1, +\infty}^- \\ I_{s_1, s_2}^{-L_1} \downarrow & \searrow I_{s_1, s_2}^{-L_1 \cup +L_2} & \downarrow I_{s_1, +\infty}^{-L_1} & & \downarrow I_{s_1, +\infty}^{-L_1} \\ A_{-\infty, s_2}^- & \xrightarrow{I_{-\infty, s_2}^{+L_2}} & A_{-\infty, +\infty}^- & \xrightarrow{id} & A_{-\infty, +\infty}^- \\ D_{-\infty, s_2}^{-L_1} \downarrow & \searrow D_{-\infty, +\infty}^{-L_1} & \downarrow D_{-\infty, +\infty}^{-L_1} & & \downarrow D_{-\infty, +\infty}^{-L_1} \\ A_{+\infty, s_2 + lk}^- & \xrightarrow{I_{+\infty, s_2 + lk}^{+L_2}} & A_{+\infty, +\infty}^- & \xrightarrow{id} & A_{+\infty, +\infty}^- \end{array} & & \begin{array}{ccccc} A_{s_1, s_2}^- & \xrightarrow{I_{s_1, s_2}^{+L_2}} & A_{s_1, +\infty}^- & \xrightarrow{id} & A_{s_1, +\infty}^- \\ I_{s_1, s_2}^{+L_1} \downarrow & \searrow I_{s_1, s_2}^{+L_1 \cup +L_2} & \downarrow I_{s_1, +\infty}^{+L_1} & & \downarrow I_{s_1, +\infty}^{+L_1} \\ A_{+\infty, s_2}^- & \xrightarrow{I_{+\infty, s_2}^{+L_2}} & A_{+\infty, +\infty}^- & \xrightarrow{id} & A_{+\infty, +\infty}^- \\ id \downarrow & \searrow id & \downarrow id & & \downarrow id \\ A_{+\infty, s_2}^- & \xrightarrow{I_{+\infty, s_2}^{+L_2}} & A_{+\infty, +\infty}^- & \xrightarrow{id} & A_{+\infty, +\infty}^- \end{array} \end{array}$$

The squares $R_{s,i,j}$'s used in Equations (4.2) to (4.5) are defined to be the compressions of $\mathfrak{R}_{s,i,j}$'s.

5. APPLYING THE SURGERY FORMULA TO TWO-BRIDGE LINKS

In this section, we show some algebraic rigidity results for the chain maps between certain chain complexes up to chain homotopy. This provides a way to determine the destabilization maps in the surgery complex of two-bridge links up to chain homotopy. Using these maps to replace the original maps in the surgery complex, we construct a *perturbed surgery complex*. We further show that it has the same homology as the original one. Based on the perturbed surgery formula, we give an algorithm for computing the homology of surgeries on two-bridge links.

5.1. Algebraic rigidity results. There is a short exact sequence in the Exercise 3.6.1 in [18] as follows. Suppose P_*, Q^* are (co)chain complexes of R -modules, and $P, d(P) = \text{Im}(d)$ are both projective R -modules. Then there is an exact sequence

$$0 \rightarrow \prod_{p+q=n-1} \text{Ext}_R^1(H_p(P_*), H^q(Q^*)) \rightarrow H_n(\text{Hom}(P_*, Q^*)) \rightarrow \prod_{p+q=n} \text{Hom}_R(H_p(P_*), H^q(Q^*)) \rightarrow 0.$$

For completeness, we give a proof here adapted to the setting of $\mathbb{Z}/2\mathbb{Z}$ -graded chain complexes.

Lemma 5.1. *Let P_*, Q_* be $\mathbb{Z}/2\mathbb{Z}$ -graded chain complexes of R -modules. If $P, d(P) = \text{Im}(d)$ are free modules, then there is a short exact sequence for any $n, p, q \in \mathbb{Z}/2\mathbb{Z}$,*

$$(5.1) \quad 0 \rightarrow \bigoplus_{p+q=n+1} \text{Ext}_R^1(H_p(P), H_q(Q)) \rightarrow H_n(\text{Hom}(P, Q)) \rightarrow \bigoplus_{p+q=n} \text{Hom}_R(H_p(P), H_q(Q)) \rightarrow 0.$$

Proof. First, all the indices n, p, q, i, j are in $\mathbb{Z}/2\mathbb{Z}$. Since $d(P)$ is free, the short exact sequence $0 \rightarrow Z^P \rightarrow P \rightarrow d(P) \rightarrow 0$ splits, thus giving that $P = d(P) \oplus Z^P$. Thereby, Z^P is projective and thus $\text{Ext}_R^1(Z^P, M) = 0$ for all R -module M . Also, by $\text{Ext}_R^1(d(P), M) = 0, \forall M$ we get an exact sequence

$$0 \rightarrow \text{Hom}_R(d(P_p), Q_q) \rightarrow \text{Hom}_R(P_p, Q_q) \rightarrow \text{Hom}_R(Z_p^P, Q_q) \rightarrow 0.$$

These assemble to a short exact sequence of chain complexes

$$(5.2) \quad 0 \rightarrow \bigoplus_{p+q=n} \text{Hom}_R(d(P_p), Q_q) \rightarrow (\text{Hom}_R(P, Q))_n \rightarrow \bigoplus_{p+q=n} \text{Hom}_R(Z_p^P, Q_q) \rightarrow 0.$$

Actually, it is not hard to check the following commuting diagram

$$\begin{array}{ccccc} 0 \rightarrow \bigoplus_{p+q=n} \text{Hom}_R(d(P_p), Q_q) & \longrightarrow & (\text{Hom}_R(P, Q))_n & \longrightarrow & \bigoplus_{p+q=n} \text{Hom}_R(Z_p^P, Q_q) \rightarrow 0 \\ & d \downarrow & d \downarrow & & d \downarrow \\ 0 \rightarrow \bigoplus_{p+q=n+1} \text{Hom}_R(d(P_p), Q_q) & \rightarrow & (\text{Hom}_R(P, Q))_{n+1} & \rightarrow & \bigoplus_{p+q=n+1} \text{Hom}_R(Z_p^P, Q_q) \rightarrow 0 \end{array}$$

Since dP is free, the short exact sequence $0 \rightarrow d(Q_{j-1}) \rightarrow Z_j^Q \rightarrow H_j(Q) \rightarrow 0$ gives a short exact sequence

$$0 \rightarrow \text{Hom}_R(dP_i, d(Q_{j-1})) \rightarrow \text{Hom}_R(dP_i, Z_j^Q) \rightarrow \text{Hom}_R(dP_i, H_j(Q)) \rightarrow 0.$$

Furthermore, the differential in $\text{Hom}_R(dP_i, Q_j)$ is dQ , so from the above exact sequence it follows that $H_n(\text{Hom}_R(d(P), Q)) = \bigoplus_{p+q=n} \text{Hom}_R(d(P_p), H_q(Q)), p, q, n \in \mathbb{Z}/2\mathbb{Z}$. Since Z^P is projective, similarly we have

$$H_n(\text{Hom}_R(Z^P, Q)) = \bigoplus_{p+q=n} \text{Hom}_R(Z_p^P, H_q(Q)).$$

Thus the long exact sequence of homology from Equation (5.2) is

$$(5.3) \quad \begin{array}{l} \cdots \rightarrow \bigoplus_{p+q=n} \text{Hom}_R(Z_{p+1}^P, H_q(Q)) \xrightarrow{\partial_{n+1}} \bigoplus_{p+q=n} \text{Hom}_R(d(P_p), H_q(Q)) \rightarrow H_n(\text{Hom}_R(P, Q)) \\ \rightarrow \bigoplus_{p+q=n} \text{Hom}_R(Z_p^P, H_q(Q)) \xrightarrow{\partial_n} \bigoplus_{p+q=n} \text{Hom}_R(d(P_{p+1}), H_q(Q)) \rightarrow \cdots \end{array}$$

A diagram chasing shows that the connecting morphism $\partial_* : \text{Hom}(Z_*^P, H_*(Q)) \rightarrow \text{Hom}(d(P_*), H_*(Q))$ is the restriction.

Hence, the short exact sequence $0 \rightarrow dP_{i+1} \rightarrow Z_i^P \rightarrow H_i(P) \rightarrow 0$ can produce the exact sequence

$$\begin{array}{l} 0 \rightarrow \text{Hom}_R(H_p(P), H_q(Q)) \rightarrow \text{Hom}_R(Z_p^P, H_q(Q)) \xrightarrow{\partial_{p+q}} \text{Hom}_R(dP_{p+1}, H_q(Q)) \\ \rightarrow \text{Ext}_R^1(H_p(P), H_q(Q)) \rightarrow \text{Ext}_R^1(Z_p^P, H_q(Q)) = 0, \end{array}$$

thus $\text{Ker}(\partial_{p+q}) \cong \text{Hom}_R(H_p(P), H_q(Q))$ and $\text{Coker}(\partial_{p+q}) \cong \text{Ext}_R^1(H_p(P), H_q(Q))$. Finally, the exact sequence in Equation (5.1) comes from Equation (5.3). \square

Let (C_*, ∂_*) be a chain complex of \mathbb{F} -vector spaces, with U_1, U_2 -actions which drop the \mathbb{Z} -grading by 2. Consider C as a $\mathbb{F}[[U_1, U_2]]$ -module. Even though the U_1, U_2 -actions do not preserve the \mathbb{Z} -grading, we will still call C a *complex of $\mathbb{F}[[U_1, U_2]]$ -modules*.

Proposition 5.2. *Let A, B be complexes of $\mathbb{F}[[U_1, U_2]]$ -modules with U_1, U_2 -actions dropping grading by 2, and $A, d(A)$ are both free $\mathbb{F}[[U_1, U_2]]$ -modules. Suppose $H_*(A) \cong H_*(B) \cong \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$, precisely, $H_{2k}(A) \cong H_{2k}(B) \cong \mathbb{F}$ for all $k \leq 0$ and $H_i(A) = H_i(B) = 0$ otherwise, where $U_i \cdot H_{2k}(A) = H_{2k-2}(A), U_i \cdot H_{2k}(B) = H_{2k-2}(B)$ for both $i = 1, 2$. If $F, G : A \rightarrow B$ are both quasi-isomorphisms as $\mathbb{F}[[U_1, U_2]]$ -modules, then F and G are homotopic as $\mathbb{F}[[U_1, U_2]]$ -modules.*

Proof. First, the \mathbb{Z} -grading of A, B induces a $\mathbb{Z}/2\mathbb{Z}$ -grading on both A and B , and U_1, U_2 -action preserves the induced $\mathbb{Z}/2\mathbb{Z}$ -grading, thus we regard A, B as $\mathbb{Z}/2\mathbb{Z}$ -graded chain complexes of $\mathbb{F}[[U_1, U_2]]$ -modules. In order to distinguish these two gradings, we put brackets on the numbers to represent $\mathbb{Z}/2\mathbb{Z}$ -gradings. Hence we have $H_{[0]}(A) = H_{[0]}(B) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2), H_{[1]}(A) = H_{[1]}(B) = 0$.

By Lemma 5.1, we have

$$\begin{aligned} 0 &\rightarrow \bigoplus_{[p] \in \mathbb{Z}/2\mathbb{Z}} \text{Ext}_{\mathbb{F}[[U_1, U_2]]}^1(H_{[p+1]}(A_*), H_{[p]}(B_*)) \rightarrow H_{[0]}(\text{Hom}(A_*, B_*)) \\ &\rightarrow \bigoplus_{[p] \in \mathbb{Z}/2\mathbb{Z}} \text{Hom}_{\mathbb{F}[[U_1, U_2]]}(H_{[p]}(A_*), H_{[p]}(B_*)) \rightarrow 0, \end{aligned}$$

thus $H_{[0]}(\text{Hom}(A_*, B_*)) = \text{Hom}_{\mathbb{F}[[U_1, U_2]]}(\mathbb{F}[[U_1, U_2]]/(U_1 - U_2), \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$. Since $H_{[0]}(\text{Hom}(A_*, B_*))$ is the group of chain homotopy equivalence classes of chain maps from A to B , this means the chain maps from A to B are classified by their action on homology. Since F and G are both quasi-isomorphisms, they are homotopic as $\mathbb{F}[[U_1, U_2]]$ -modules.

Let $H : A \rightarrow B$ be any homotopy such that $F - G = H\partial + \partial H, H \cdot U_i = U_i \cdot H$. Then H shifts the $\mathbb{Z}/2\mathbb{Z}$ -grading by 1, thus shifting the \mathbb{Z} -grading by odd numbers. Thus, let $H = \sum_{i \in \mathbb{Z}} H_{2i+1}$, where $H_{2i+1} : A_* \rightarrow B_{*+2i+1}$. Since

$$F - G = H\partial + \partial H = \sum_{i \in \mathbb{Z}} (H_{2i+1}\partial + \partial H_{2i+1})$$

preserves the original \mathbb{Z} -grading, we have $H_{2i+1}\partial + \partial H_{2i+1} = 0, \forall i \neq 0$. So we can replace the homotopy H by $H_1 : A_* \rightarrow B_{*+1}$, thus being a chain homotopy of the original \mathbb{Z} -graded chain complexes. \square

Corollary 5.3. *Suppose the complexes A_* and B_* are as in Proposition 5.2, then A_* and B_* are homotopy equivalent.*

Proof. From the proof of Proposition 5.2, we see $H_{[0]}(\text{Hom}(A_*, B_*)) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$, which implies that there exists a quasi-isomorphism $h : A \rightarrow B$ as $\mathbb{Z}/2\mathbb{Z}$ -graded chain complex of $\mathbb{F}[[U_1, U_2]]$ -modules. Decompose h as $h = h_0 + h_1$ such that for all $a \in A_n$, $h_0(a) \in B_n, h_1(a) \in \oplus_{i \neq 0} B_{n+2i}$. Then $h\partial_A = \partial_B h$ implies that $h_0\partial_A = \partial_B h_0, h_1\partial_A = \partial_B h_1$, so h_0 is also a chain map preserving the \mathbb{Z} -grading. Since $U_i h = h U_i, \forall i = 1, 2$, and U_i -action drops by 2, thus $h_0 U_i + U_i h_0 = 0$. In addition, h_0 is also a quasi-isomorphism. Hence, on the homology level, $h_0(1) = 1 \in \mathbb{F}[[U_1, U_2]]/U_1 - U_2$.

Similarly, we have another quasi-isomorphism $g_0 : B_* \rightarrow A_*$ preserving the \mathbb{Z} -grading, such that $g\partial_B = \partial_A g, gU_i = U_i g$. Then, $g_0 f_0 : A_* \rightarrow A_*$ is a quasi-isomorphism, from Proposition 5.2, we get that $g_0 f_0 - id_A = \partial H + H\partial$, where H is chain homotopy of \mathbb{Z} -graded complexes commuting with U_i -action. Similarly, we have $f_0 g_0$ is homotopic to id_B . \square

Proposition 5.4. *Let A_*, B_* be complexes of $\mathbb{F}[[U]]$ -modules with U -action dropping grading by 2. Suppose $H_*(A) = H_*(B) = \mathbb{F}[[U]]$, precisely, $H_{2k}(A) \cong H_{2k}(B) \cong \mathbb{F}$ for all $k \leq 0$ and $H_i(A) = H_i(B) = 0$ otherwise, where $U \cdot H_{2k}(A) = H_{2k-2}(A), U \cdot H_{2k}(B) = H_{2k-2}(B)$. If H, K are both*

chain homotopies as homomorphisms of $\mathbb{F}[[U]]$ -modules between two chain maps $F, G : A \rightarrow B$, i.e. $H\partial + \partial H = K\partial + \partial K = F - G$, then $H - K = \partial T + T\partial$, for some $\mathbb{F}[[U]]$ -module homomorphism $T : A_* \rightarrow B_{*+2}$.

Proof. First, we regard A, B as $\mathbb{Z}/2\mathbb{Z}$ -graded complexes of $\mathbb{F}[[U]]$ -modules. By Lemma 5.1, we can compute $H_{[1]}(\text{Hom}(A, B)) = \text{Ext}_{\mathbb{F}[[U]]}^1(\mathbb{F}[[U]], \mathbb{F}[[U]]) = 0$. Since $\partial(H - K) + (H - K)\partial = 0$, then $H - K \in Z_{[1]}(\text{Hom}(A, B)) = B_{[1]}(\text{Hom}(A, B))$, this means there is a homomorphism of $\mathbb{F}[[U]]$ -modules $T : A \rightarrow B$ preserving the $\mathbb{Z}/2\mathbb{Z}$ -grading, such that $H - K = \partial T + T\partial$. Thus, the map T can be decomposed as $T = \sum_{i \in \mathbb{Z}} T_{2i}$, where $T_{2i} : A_* \rightarrow B_{*+2i}$. From the fact that $H - K = \sum_{i \in \mathbb{Z}} \partial T_{2i} + T_{2i}\partial$ maps A_n into B_{n+1} , it follows that $\partial T_{2i} + T_{2i}\partial : A_* \rightarrow B_{*+2i-1}$ vanish for all $i \neq 1$. Thus $T = T_2 : A_* \rightarrow B_{*+2}$. \square

5.2. Destabilization maps. In the link surgery formula, the part of polygon counts for defining the destabilization maps is difficult to read off from the Heegaard diagram, and higher polygon counts usually depend on the almost complex structure of Σ . However, in the case of two-bridge links we can use the algebraic rigidity result to avoid the difficulty.

In a primitive system of hyperboxes, all the destabilization maps we need are listed below:

$$\begin{aligned} D_{-\infty, s_2}^{-L_1} &: \mathfrak{A}(\mathcal{H}^L, -\infty, s_2) \rightarrow \mathfrak{A}^-(\mathcal{H}^L, +\infty, s_2 + \text{lk}(L_1, L_2)), \\ D_{s_1, -\infty}^{-L_2} &: \mathfrak{A}(\mathcal{H}^L, s_1, -\infty) \rightarrow \mathfrak{A}^-(\mathcal{H}^{L_2}, s_1 + \text{lk}(L_1, L_2), +\infty), \\ D_{-\infty, -\infty}^{-L_1 \cup -L_2} &: \mathfrak{A}^-(\mathcal{H}^L, -\infty, -\infty) \rightarrow \mathfrak{A}^-(\mathcal{H}^L, +\infty, +\infty). \end{aligned}$$

Second, notice that all the domains and targets of these maps have homology $\mathbb{F}[[U_1, U_2]]/U_1 - U_2$. By Proposition 5.2, we can substitute $D_{-\infty, s_2}^{-L_1}, D_{s_1, -\infty}^{-L_2}$ by any homotopy equivalence, since they are all homotopic as homomorphisms of $\mathbb{F}[[U_1, U_2]]$ -modules. By Proposition 5.4, we can also substitute the diagonal maps $D_{-\infty, -\infty}^{-L_1 \cup -L_2}$ by any $\mathbb{F}[[U_1]]$ -linear homotopy shifting grading by 1, since they are homotopic up to higher homotopy as homomorphisms of $\mathbb{F}[[U_1]]$ -modules. We will show an invariance theorem of the surgery square under perturbations of the edge maps and the diagonal maps in the next sections.

5.3. Perturbed surgery complex for two-bridge links. In the previous section, the rigidity results allow us to perturb the edges and diagonal maps up to homotopies, in the surgery square for two-bridge links. However, in order to obtain a square of chain complexes, we still need some more modifications of the square.

First, suppose we have a hypercube $(C^\varepsilon, D^\varepsilon)$. If we change $D_{\varepsilon_0}^\varepsilon$ to $D_{\varepsilon_0}^{\varepsilon'} = D_{\varepsilon_0}^\varepsilon + \Delta D_{\varepsilon_0}^\varepsilon$ for all ε with $\|\varepsilon\| > 0$, then in order to have a hypercube again, we need to have

$$\begin{aligned} \sum_{\varepsilon' \leq \varepsilon} (D_{\varepsilon_0 + \varepsilon'}^{\varepsilon - \varepsilon'} + \Delta D_{\varepsilon_0 + \varepsilon'}^{\varepsilon - \varepsilon'}) \circ (D_{\varepsilon_0}^{\varepsilon'} + \Delta D_{\varepsilon_0}^{\varepsilon'}) &= 0, \\ \sum_{\varepsilon' \leq \varepsilon} \Delta D_{\varepsilon_0 + \varepsilon'}^{\varepsilon - \varepsilon'} \circ D_{\varepsilon_0}^{\varepsilon'} + D_{\varepsilon_0 + \varepsilon'}^{\varepsilon - \varepsilon'} \circ \Delta D_{\varepsilon_0}^{\varepsilon} + \Delta D_{\varepsilon_0 + \varepsilon'}^{\varepsilon - \varepsilon'} \circ \Delta D_{\varepsilon_0}^{\varepsilon} &= 0. \end{aligned}$$

This formula provides a necessary condition to inductively perturb the maps from edges to the longest diagonal. Based on the above principles, we get the following procedures to construct the perturbed surgery square.

Suppose \mathcal{H} be a primitive system of hyperboxes of a two-bridge link L and consider Equation (4.12). Now we choose an arbitrary homotopy equivalence $\tilde{D}_{s_1, s_2}^{-L_i}$ as $\mathbb{F}[[U_1, U_2]]$ -modules for substituting $D_{s_1, s_2}^{-L_i}$. By Proposition 5.2, $\tilde{D}_{s_1, s_2}^{-L_i}$ and $D_{s_1, s_2}^{-L_i}$ are homotopic by a $\mathbb{F}[[U_1, U_2]]$ -linear homotopy $H_{s_1, s_2}^{-L_i}$:

$$\tilde{D}_s^{-L_i} = D_s^{-L_i} + H_s^{-L_i} \partial_s^- + \partial_{p^{L_i}(s)}^- H_s^{-L_i}.$$

Then, we choose any $\mathbb{F}[[U_1]]$ -linear maps $\tilde{F}_{s_1, -\infty}^{\pm L_1 \cup -L_2}, \tilde{F}_{-\infty, s_2}^{\pm L_1 \cup \pm L_2}, \tilde{D}_{-\infty, -\infty}^{-L_1 \cup -L_2}$ which are homotopies in each square of Equation (4.12), such that the following rectangles are hyperboxes of chain complexes:

(5.4)

$$\begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{I_{s_1, s_2}^{-L_2}} & A_{s_1, -\infty}^- \xrightarrow{\tilde{D}_{s_1, -\infty}^{-L_2}} A_{s_1 + \text{lk}, +\infty}^- \\
 \downarrow I_{s_1, s_2}^{-L_1} & \searrow I_{s_1, s_2}^{-L_1 \cup -L_2} & \downarrow I_{s_1, -\infty}^{-L_1} \quad \tilde{F}_{s_1, -\infty}^{-L_1 \cup -L_2} \searrow \downarrow I_{s_1 + \text{lk}, +\infty}^{-L_1} \\
 A_{-\infty, s_2}^- & \xrightarrow{I_{-\infty, s_2}^{-L_2}} & A_{-\infty, -\infty}^- \xrightarrow{\tilde{D}_{-\infty, -\infty}^{-L_2}} A_{-\infty, +\infty}^- \\
 \downarrow \tilde{D}_{-\infty, s_2}^{-L_1} & \searrow \tilde{F}_{-\infty, s_2}^{-L_1 \cup -L_2} & \downarrow \tilde{D}_{-\infty, -\infty}^{-L_1} \quad \tilde{D}_{-\infty, -\infty}^{-L_1 \cup -L_2} \searrow \downarrow \tilde{D}_{-\infty, +\infty}^{-L_1} \\
 A_{+\infty, s_2 + \text{lk}}^- & \xrightarrow{I_{+\infty, s_2 + \text{lk}}^{-L_2}} & A_{+\infty, -\infty}^- \xrightarrow{\tilde{D}_{+\infty, -\infty}^{-L_2}} A_{+\infty, +\infty}^-;
 \end{array}
 \quad
 \begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{I_{s_1, s_2}^{-L_2}} & A_{s_1, -\infty}^- \xrightarrow{\tilde{D}_{s_1, -\infty}^{-L_2}} A_{s_1 + \text{lk}, +\infty}^- \\
 \downarrow I_{s_1, s_2}^{+L_1} & \searrow I_{s_1, s_2}^{+L_1 \cup -L_2} & \downarrow I_{s_1, -\infty}^{+L_1} \quad \tilde{F}_{s_1, -\infty}^{+L_1 \cup -L_2} \searrow \downarrow I_{s_1 + \text{lk}, +\infty}^{+L_1} \\
 A_{+\infty, s_2}^- & \xrightarrow{I_{+\infty, s_2}^{-L_2}} & A_{+\infty, -\infty}^- \xrightarrow{\tilde{D}_{+\infty, -\infty}^{-L_2}} A_{+\infty, +\infty}^- \\
 \downarrow id & \searrow id & \downarrow id \\
 A_{+\infty, s_2}^- & \xrightarrow{I_{+\infty, s_2}^{-L_2}} & A_{+\infty, -\infty}^- \xrightarrow{\tilde{D}_{+\infty, -\infty}^{-L_2}} A_{+\infty, +\infty}^-;
 \end{array}$$

$$\begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{I_{s_1, s_2}^{+L_2}} & A_{s_1, +\infty}^- \xrightarrow{id} A_{s_1, +\infty}^- \\
 \downarrow I_{s_1, s_2}^{-L_1} & \searrow I_{s_1, s_2}^{-L_1 \cup +L_2} & \downarrow I_{s_1, +\infty}^{-L_1} \quad \tilde{F}_{s_1, +\infty}^{-L_1 \cup +L_2} \searrow \downarrow I_{s_1, +\infty}^{-L_1} \\
 A_{-\infty, s_2}^- & \xrightarrow{I_{-\infty, s_2}^{+L_2}} & A_{-\infty, +\infty}^- \xrightarrow{id} A_{-\infty, +\infty}^- \\
 \downarrow \tilde{D}_{-\infty, s_2}^{-L_1} & \searrow \tilde{F}_{-\infty, s_2}^{-L_1 \cup +L_2} & \downarrow \tilde{D}_{-\infty, +\infty}^{-L_1} \quad \tilde{D}_{-\infty, +\infty}^{-L_1 \cup +L_2} \searrow \downarrow \tilde{D}_{-\infty, +\infty}^{-L_1} \\
 A_{+\infty, s_2 + \text{lk}}^- & \xrightarrow{I_{+\infty, s_2}^{+L_2}} & A_{+\infty, +\infty}^- \xrightarrow{id} A_{+\infty, +\infty}^-;
 \end{array}
 \quad
 \begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{I_{s_1, s_2}^{+L_2}} & A_{s_1, +\infty}^- \xrightarrow{id} A_{s_1, +\infty}^- \\
 \downarrow I_{s_1, s_2}^{+L_1} & \searrow I_{s_1, s_2}^{+L_1 \cup +L_2} & \downarrow I_{s_1, +\infty}^{+L_1} \quad \tilde{F}_{s_1, +\infty}^{+L_1 \cup +L_2} \searrow \downarrow I_{s_1, +\infty}^{+L_1} \\
 A_{+\infty, s_2}^- & \xrightarrow{I_{+\infty, s_2}^{+L_2}} & A_{+\infty, +\infty}^- \xrightarrow{id} A_{+\infty, +\infty}^- \\
 \downarrow id & \searrow id & \downarrow id \\
 A_{+\infty, s_2}^- & \xrightarrow{I_{+\infty, s_2}^{+L_2}} & A_{+\infty, +\infty}^- \xrightarrow{id} A_{+\infty, +\infty}^-.
 \end{array}$$

Definition 5.5 (Perturbed surgery square). The above rectangles in Equation (5.4) are called *perturbed surgery rectangles* for two-bridge links. After compressing them, we get four sets of squares,

$$\begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{\tilde{\Phi}_{s_1, s_2}^{-L_2}} & A_{s_1 + \text{lk}, +\infty}^- \\
 \downarrow \tilde{\Phi}_{s_1, s_2}^{-L_1} & \searrow \tilde{\Phi}_{s_1, s_2}^{-L_1 \cup -L_2} & \downarrow \tilde{\Phi}_{s_1 + \text{lk}, +\infty}^{-L_1} \\
 A_{+\infty, s_2 + \text{lk}}^- & \xrightarrow{\tilde{\Phi}_{+\infty, s_2 + \text{lk}}^{-L_2}} & A_{+\infty, +\infty}^-;
 \end{array}
 \quad
 \begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{\tilde{\Phi}_{s_1, s_2}^{+L_2}} & A_{s_1, +\infty}^- \\
 \downarrow \tilde{\Phi}_{s_1, s_2}^{-L_1} & \searrow \tilde{\Phi}_{s_1, s_2}^{-L_1 \cup +L_2} & \downarrow \tilde{\Phi}_{s_1, +\infty}^{-L_1} \\
 A_{+\infty, s_2 + \text{lk}}^- & \xrightarrow{\tilde{\Phi}_{+\infty, s_2 + \text{lk}}^{+L_2}} & A_{+\infty, +\infty}^-;
 \end{array}$$

$$\begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{\tilde{\Phi}_{s_1, s_2}^{-L_2}} & A_{s_1 + \text{lk}, +\infty}^- \\
 \downarrow \tilde{\Phi}_{s_1, s_2}^{+L_1} & \searrow \tilde{\Phi}_{s_1, s_2}^{+L_1 \cup -L_2} & \downarrow \tilde{\Phi}_{s_1 + \text{lk}, +\infty}^{+L_1} \\
 A_{+\infty, s_2}^- & \xrightarrow{\tilde{\Phi}_{+\infty, s_2}^{-L_2}} & A_{+\infty, +\infty}^-;
 \end{array}
 \quad
 \begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{\tilde{\Phi}_{s_1, s_2}^{+L_2}} & A_{s_1, +\infty}^- \\
 \downarrow \tilde{\Phi}_{s_1, s_2}^{+L_1} & \searrow \tilde{\Phi}_{s_1, s_2}^{+L_1 \cup +L_2} & \downarrow \tilde{\Phi}_{s_1, +\infty}^{+L_1} \\
 A_{+\infty, s_2}^- & \xrightarrow{\tilde{\Phi}_{+\infty, s_2}^{+L_2}} & A_{+\infty, +\infty}^-.
 \end{array}$$

After a Λ -twisted gluing of the above squares, we obtain a *perturbed surgery square* $(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{D}^-)$.

Remark 5.6. In the definition, a perturbed surgery square depends on the choices of the maps $\tilde{D}_{s_1, s_2}^{-L_i}, \tilde{F}_{s_1, -\infty}^{\pm L_1 \cup -L_2}, \tilde{F}_{-\infty, s_2}^{\pm L_1 \cup \pm L_2}, \tilde{D}_{-\infty, -\infty}^{-L_1 \cup -L_2}$. However, we will show it is isomorphic to the original square as $\mathbb{F}[[U_1]]$ -module.

5.4. Invariance of the perturbed surgery complex. Now we establish the invariance of the perturbed surgery complex for two-bridge links under the change of edge maps and some diagonal maps up to chain homotopies.

Proposition 5.7. *Let K be a field with $\text{char}(K) = 2$ and R be a K -algebra. Suppose $f, g : A \rightarrow B$ be two chain maps between two chain complexes of R -modules. If f, g are homotopic to each other by $f \xrightarrow{H} g$, then the mapping cones $\text{cone}(f), \text{cone}(g)$ are isomorphic.*

Proof. We directly construct the isomorphism between the mapping cones $\text{cone}(f)$ and $\text{cone}(g)$. Define $K_1 : \text{cone}(f) \rightarrow \text{cone}(g), K_2 : \text{cone}(g) \rightarrow \text{cone}(f)$ by

$$\begin{aligned} K_1|_A &= id_A \oplus H, K_1|_B = id_B, \\ K_2|_A &= id_A \oplus H, K_2|_B = id_B. \end{aligned}$$

In fact, K_1 is a chain map, since $\forall a \in A, b \in B$,

$$\begin{aligned} K_1 \partial_f(a) + \partial_g K_1(a) &= K_1(\partial_A(a) + f(a)) + \partial_g(a + H(a)) \\ &= \partial(a) + H \partial_A(a) + f(a) + \partial_A(a) + g(a) + \partial_B H(a) = 0, \\ K_1 \partial_f(b) + \partial_g K_1(b) &= K_1 \partial_B(b) + \partial_g(b) = \partial_B(b) + \partial_B(b) = 0. \end{aligned}$$

Moreover, $K_2 K_1$ is $id_{\text{cone}(f)}$, since

$$\begin{aligned} K_2 K_1(a) &= K_2(a + H(a)) = a + H(a) + H(a) = a, \\ K_2 K_1(b) &= K_2(b) = b, \forall a \in A, b \in B. \end{aligned}$$

$$\begin{array}{ccccc} A & \xrightarrow{id_A} & A & \xrightarrow{id_A} & A \\ f \downarrow & \searrow H & g \downarrow & \searrow H & f \downarrow \\ B & \xrightarrow{id_B} & B & \xrightarrow{id_B} & B \end{array}$$

□

There is a hyperbox version of Proposition 5.7.

Definition 5.8. A hyperbox of chain complexes H is said to be *isomorphic* to another hyperbox H' , if there are chain maps of hyperboxes $F : H \rightarrow H', G : H' \rightarrow H$, such that $F \circ G = id_{H'}, G \circ F = id_H$.

Proposition 5.9. *Let $H = ((C^\varepsilon)_{\varepsilon \in \mathbb{E}((d,1))}, (D^\varepsilon)_{\varepsilon \in \mathbb{E}(n+1)})$ be a hyperbox of chain complexes of size $(\mathbf{d}, 1) \in \mathbb{Z}_{\geq 0}^{n+1}$. If all the edge maps $D^{(\mathbf{0},1)} = id$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^n$, then H induces an isomorphism from the subhyperbox $C^{\varepsilon_{n+1}=0}$ to the subhyperbox $C^{\varepsilon_{n+1}=1}$.*

Proof. By induction we first show the case of hypercubes, i.e., $\mathbf{d} = (1, \dots, 1) \in \mathbb{Z}^n$.

When $n = 1$, this is exactly Proposition 5.8. When $n > 2$, first we make some notations. There is a $(n-1)$ -dimensional subhypercube corresponding to $\varepsilon_n = \varepsilon_{n+1} = 0$, denoted by C^{00} , and there is also a $(n-1)$ -dimensional subhypercube corresponding to $\varepsilon_n = 0, \varepsilon_{n+1} = 1$, denoted by C^{01} . Similarly, the subhypercube corresponding to $\varepsilon_n = 1, \varepsilon_{n+1} = 0$ is denoted by C^{10} , and the hypercube corresponding to $\varepsilon_n = \varepsilon_{n+1} = 1$ is denoted by C^{11} . Then we can view the hypercube H as the following square of hypercubes.

$$\begin{array}{ccc} C^{00} & \xrightarrow{f} & C^{10} \\ h_1 \downarrow & \searrow H & \downarrow h_2 \\ C^{01} & \xrightarrow{f'} & C^{11} \end{array}$$

Notice that f, f', h_1, h_2 are chain maps of hypercubes, and H is a chain homotopy of hypercubes between the chain maps. In other words, we have

$$h_1 \circ D|_{C^{00}} = D|_{C^{10}} \circ h_1, h_2 \circ D|_{C^{01}} = D|_{C^{11}} \circ h_2, H \circ D|_{C^{00}} + D|_{C^{11}} \circ H = h_2 \circ f + f' \circ h_1.$$

By induction, the n -dimensional subhypercube corresponding to $\varepsilon_n = 0$ induces the isomorphism h_1 . Thus, we have a chain map of hypercubes $h_1^{-1} : C^{01} \rightarrow C^{00}$, such that $h_1 h_1^{-1} = id_{C^{01}}$ and $h_1^{-1} h_1 = id_{C^{00}}$. Similarly, we have $h_2^{-1} : C^{11} \rightarrow C^{10}$ as the inverse of h_2 . The hypercube H induces a chain map $h_1 + H + h_2$ from the subhypercube $C^{\varepsilon_{n+1}=0}$ corresponding to $\varepsilon_{n+1} = 0$ to the subhypercube $C^{\varepsilon_{n+1}=1}$ corresponding to $\varepsilon_{n+1} = 1$. We show that the chain map of hyperboxes $h_1 + H + h_2 : C^{\varepsilon_{n+1}=0} \rightarrow C^{\varepsilon_{n+1}=1}$ is an isomorphism by directly constructing the inverse $h_1^{-1} + h_2^{-1} \circ H \circ h_1^{-1} + h_2^{-1} : C^{\varepsilon_{n+1}=1} \rightarrow C^{\varepsilon_{n+1}=0}$ which is induced by the following hypercube:

$$\begin{array}{ccc} C^{01} & \xrightarrow{f'} & C^{11} \\ h_1^{-1} \downarrow & \searrow h_2^{-1} \circ H \circ h_1^{-1} & \downarrow h_2^{-1} \\ C^{00} & \xrightarrow{f} & C^{10} \end{array}$$

Here, the map $h_2^{-1} \circ H \circ h_1^{-1}$ is the ordinary composition as maps of h_2^{-1}, H, h_1^{-1} .

The following two rectangles of hypercubes show that $h_1^{-1} + h_2^{-1} H h_1^{-1} + h_2^{-1}$ is the inverse of $h_1 + H + h_2$.

$$\begin{array}{ccc} C^{00} & \xrightarrow{f} & C^{10} \\ h_1 \downarrow & \searrow H & \downarrow h_2 \\ C^{01} & \xrightarrow{f'} & C^{11} \\ h_1^{-1} \downarrow & \searrow h_2^{-1} H h_1^{-1} & \downarrow h_2^{-1} \\ C^{00} & \xrightarrow{f} & C^{10} \end{array} \quad \begin{array}{ccc} C^{01} & \xrightarrow{f'} & C^{11} \\ h_1^{-1} \downarrow & \searrow h_2^{-1} H h_1^{-1} & \downarrow h_2^{-1} \\ C^{00} & \xrightarrow{f} & C^{10} \\ h_1 \downarrow & \searrow H & \downarrow h_2 \\ C^{01} & \xrightarrow{f'} & C^{11} \end{array}$$

Fixing n , we do induction on the size of H to prove the general case for hyperboxes. We show that for all $1 \leq k \leq n-1$, if the proposition is true for any hyperbox H of size $(\mathbf{d}, 1)$ where $\mathbf{d} = (d_1, \dots, d_k, 0, 0, \dots, 0) \in \mathbb{Z}_{\geq 0}^n$, then the proposition is also true for any hyperbox of size $(\mathbf{d}', 1)$ where $\mathbf{d}' = (d'_1, \dots, d'_{k+1}, 0, \dots, 0) \in \mathbb{Z}_{\geq 0}^n$.

Let $C^{i,j} = C^{\varepsilon_{k+1}=i, \varepsilon_{n+1}=j}$, $i \in \{0, 1, \dots, d'_{k+1}\}, j \in \{0, 1\}$ be the subhyperbox corresponding to $\varepsilon_{k+1} = i, \varepsilon_{n+1} = j$. Thereby, the subhyperbox $C^{i,j}$ is of size $\mathbf{d}'_k = (d'_1, \dots, d'_k, 0, \dots, 0) \in \mathbb{Z}_{\geq 0}^{n+1}$. So we regard $C^{i,j}$ as a k -dimensional hyperbox of size $\bar{\mathbf{d}}'_k = (d'_1, \dots, d'_k)$, and for all $\varepsilon \in \mathbb{E}(\bar{\mathbf{d}}'_k)$, we denote the chain complex of H' sitting at $(\varepsilon, i, 0, 0, \dots, 0, j)$ by $(C^{i,j})^\varepsilon$.

We can decompose the hyperbox H' as a rectangle of hyperboxes as the following diagram:

$$\begin{array}{ccccccc} C^{0,0} & \xrightarrow{f_1} & C^{1,0} & \xrightarrow{f_2} & C^{2,0} & \xrightarrow{f_3} & \dots & \xrightarrow{f_{d'_{k+1}}} & C^{d'_{k+1},0} \\ \downarrow h_0 & \searrow H_1 & \downarrow h_1 & \searrow H_2 & \downarrow h_2 & \searrow H_3 & & \searrow H_{d'_{k+1}} & \downarrow h_{d'_{k+1}} \\ C^{0,1} & \xrightarrow{f'_1} & C^{1,1} & \xrightarrow{f'_2} & C^{2,1} & \xrightarrow{f'_3} & \dots & \xrightarrow{f'_{d'_{k+1}}} & C^{d'_{k+1},1} \end{array}$$

where $f_1, \dots, f_{d'_{k+1}}, f'_1, \dots, f'_{d'_{k+1}}, h_0, \dots, h_{d'_{k+1}}$ are chain maps of hyperboxes and $H_1, \dots, H_{d'_{k+1}}$ are chain homotopies of hyperboxes.

By the induction hypothesis, the subhyperbox $C^{\varepsilon_{k+1}=j}$, $j \in \{0, 1, \dots, d'_{k+1}\}$ is of size $(d'_1, \dots, d'_k, 0, \dots, 0, 1)$, and thereby induces the isomorphism of hyperboxes $h_j : C^{j,0} \rightarrow C^{j,1}$. Let the inverse of h_j be $h_j^{-1} : C^{j,1} \rightarrow C^{j,0}$. We define a set of homotopies of hyperboxes $h_j^{-1} \circ H_j \circ h_{j-1}^{-1} : C^{j-1,1} \rightarrow C^{j,0}$, for any $j \in \{1, 2, \dots, d'_{k+1}\}$ by the following equations, for all $\varepsilon^0 \in \mathbb{E}(\bar{\mathbf{d}}'_k)$, $\varepsilon \in \mathbb{E}(k)$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\bar{\mathbf{d}}'_k)$,

$$(h_j^{-1} \circ H_j \circ h_{j-1}^{-1})_{\varepsilon^0}^{\varepsilon} = \sum_{\{\varepsilon', \varepsilon'' \in \mathbb{E}(k) \mid \varepsilon' \leq \varepsilon'' \leq \varepsilon\}} (h_j^{-1})_{\varepsilon^0 + \varepsilon''}^{\varepsilon - \varepsilon''} \circ (H_j)_{\varepsilon^0 + \varepsilon'}^{\varepsilon'' - \varepsilon'} \circ (h_{j-1}^{-1})_{\varepsilon^0}^{\varepsilon'}.$$

We simply denote $h_j^{-1} \circ H_j \circ h_{j-1}^{-1}$ by $h_j^{-1} H_j h_{j-1}^{-1}$. From the definition of $h_j^{-1} H_j h_{j-1}^{-1}$, we can show the associativity of compositions of maps of hyperboxes. Thus, $H_j D|_{C^{j-1,0}} + D|_{C^{j,1}} H_j = h_j f_j + f'_j h_{j-1}$ and $h_j D|_{C^{j,0}} = D|_{C^{j,1}} h_j$ implies that

$$h_j^{-1} \circ H_j \circ h_{j-1}^{-1} \circ D|_{C^{j-1,1}} + D|_{C^{j,0}} \circ h_j^{-1} \circ H_j \circ h_{j-1}^{-1} = f_j \circ h_{j-1}^{-1} + h_j^{-1} \circ f'_j.$$

Therefore, we can construct the following the hypercube K

$$\begin{array}{ccccccc} C^{0,1} & \xrightarrow{f'_1} & C^{1,1} & \xrightarrow{f'_2} & C^{2,1} & \xrightarrow{f'_3} & \dots \xrightarrow{f'_{d'_{k+1}}} C^{d'_{k+1},1} \\ \downarrow h_0^{-1} & \searrow h_1^{-1} H_1 h_0^{-1} & \downarrow h_1^{-1} & \searrow h_2^{-1} H_2 h_1^{-1} & \downarrow h_2^{-1} & \searrow h_3^{-1} H_3 h_2^{-1} & \dots \searrow h_{d'_{k+1}}^{-1} H_{d'_{k+1}} h_{d'_{k+1}-1}^{-1} \downarrow h_{d'_{k+1}}^{-1} \\ C^{0,0} & \xrightarrow{f_1} & C^{1,0} & \xrightarrow{f_2} & C^{2,0} & \xrightarrow{f_3} & \dots \xrightarrow{f_{d'_{k+1}}} C^{d'_{k+1},0}, \end{array}$$

which induces a chain map from the subhyperbox $C^{\varepsilon_{n+1}=1}$ to the subhyperbox $C^{\varepsilon_{n+1}=0}$. Direct computations show that the chain maps induced by H and K are the inverse of each other. \square

Corollary 5.10. *On a rectangle of chain complexes $R = (C^\varepsilon, D^\varepsilon)$, if we change the diagonal maps $D_\varepsilon^{(1,1)}$ by a higher homotopy, i.e. $D_\varepsilon'^{(1,1)} = D_\varepsilon^{(1,1)} + H_\varepsilon^{(1,1)} D_\varepsilon^\varepsilon + D_{\varepsilon+(1,1)}^\varepsilon H_\varepsilon^{(1,1)}$ with $H_\varepsilon^{(1,1)} : C_*^\varepsilon \rightarrow C_{*+2}^{\varepsilon+(1,1)}$, then the new rectangle $R' = (C^\varepsilon, D'^\varepsilon)$ is isomorphic to R .*

Theorem 5.11. *Suppose L is an oriented two-bridge link with the framing Λ . For any homotopy equivalence $\tilde{D}_{s_1, s_2}^{-L_i}$ as $\mathbb{F}[[U_1, U_2]]$ -modules and $\mathbb{F}[[U_1]]$ -linear homotopies $\tilde{F}_{s_1, s_2}^{-L_1 \cup \pm L_2}$, $\tilde{F}_{s_1, s_2}^{\pm L_1 \cup -L_2}$, $\tilde{D}_{-\infty, -\infty}^{-L_1 \cup -L_2}$, the perturbed surgery formula $(\tilde{C}^-(\mathcal{H}^L, \Lambda), \tilde{D}^-)$ is isomorphic to the original surgery complex in [5] as $\mathbb{F}[[U_1]]$ -module. By imposing the U_2 -action to be the same as U_1 -action, the $\mathbb{F}[[U_1, U_2]]$ -module $H_*(\tilde{C}^-(\mathcal{H}^L, \Lambda), \tilde{D}^-)$ is isomorphic to the homology $\mathbf{HF}^-(S_\Lambda^3(L))$. Furthermore, this isomorphism preserves the grading.*

Proof. By Proposition 5.9, the following cubes show that the top square in each cube is isomorphic to the bottom one.

$$\begin{array}{ccccc} & & C & \xrightarrow{f_2} & D \\ & \nearrow f_1 + K \partial_A + \partial_C K & & \nearrow H + f_2 K & \\ A & \xrightarrow{f_3} & B & & \\ \downarrow id & \searrow K & \downarrow id & \searrow f_4 & \downarrow id \\ & & C & \xrightarrow{f_2} & D \\ & \nearrow f_1 & & \nearrow H & \\ A & \xrightarrow{f_3} & B & & \end{array} \quad \begin{array}{ccccc} & & C & \xrightarrow{f_2} & D \\ & \nearrow f_1 & & \nearrow H + K f_3 & \\ A & \xrightarrow{f_3} & B & & \\ \downarrow id & \searrow K & \downarrow id & \searrow f_4 + K \partial_B + \partial_D K & \downarrow id \\ & & C & \xrightarrow{f_2} & D \\ & \nearrow f_1 & & \nearrow H & \\ A & \xrightarrow{f_3} & B & & \end{array}$$

This means when an edge map is changed up to a chain homotopy in a square R , we are able to change the diagonal maps correspondingly to guarantee the new square is isomorphic to the original one. By inductions on the edges of rectangles in Equation (5.4), we can show that after changing the edge maps $\tilde{D}_{s_1, s_2}^{-L_i}$ up to homotopy and changing some diagonal maps accordingly, the result rectangle is isomorphic to the original one as $\mathbb{F}[[U_1, U_2]]$ -modules. In fact, we only have changed diagonal maps among $\tilde{F}_{s_1, s_2}^{\pm L_1 \cup \pm L_2}, \tilde{F}_{s_1, s_2}^{\pm L_1 \cup \pm L_2}, \tilde{D}_{\pm\infty, \pm\infty}^{\pm L_1 \cup \pm L_2}$, where we can apply the rigidity results in Proposition 5.4. Thereby, Corollary 5.10 implies the perturbed rectangles in Equation (5.4) are isomorphic to the original ones in Equation (4.12) as $\mathbb{F}[[U_1]]$ -modules. After compressing these rectangles and gluing them together, the perturbed surgery complex is isomorphic to the original surgery complex as an $\mathbb{F}[[U_1]]$ -module.

From Theorem 4.15, it follows that the U_1, U_2 actions in $H_*(\mathcal{C}^-(\mathcal{H}^L, \Lambda), \mathcal{D}^-)$ are the same. Thus by imposing the U_2 -action as the same as the U_1 -action on the $\mathbb{F}[[U_1]]$ -module $H_*(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$, we get an isomorphism as $\mathbb{F}[[U_1, U_2]]$ -module between $H_*(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$ and $H_*(\mathcal{C}^-(\mathcal{H}^L, \Lambda), \mathcal{D}^-)$. As all the rigidity results respect the grading, the above isomorphism also preserves the grading. \square

Remark 5.12. In the above theorem, the homology of the unknot is $\mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$ as an $\mathbb{F}[[U_1, U_2]]$ -module. There is no analogue of the Proposition 5.4 for homotopies over the ring $\mathbb{F}[[U_1, U_2]]$. This is why we restrict our scalars to $\mathbb{F}[[U_1]]$. This idea is due to Ciprian Manolescu.

5.5. Algorithm for computing $\mathbf{HF}^-(S_\Lambda^3(L))$. First, we use the algorithm in Section 3.4 to compute all the $A_s^-(L)$.

Second, by solving linear equations, we find quasi-homomorphisms

$$\tilde{D}_{-\infty, s_2}^{-L_1} : A_{-\infty, s_2}^- \rightarrow A_{-\infty, s_2 + \text{lk}}^-, \quad \tilde{D}_{s_1, -\infty}^{-L_2} : A_{s_1, -\infty}^- \rightarrow A_{s_1 + \text{lk}, -\infty}^-.$$

Finding chain maps is a problem of solving linear equations modulo 2, which has fast algorithm in the case of sparse matrices. Actually, from the Schubert diagram \mathcal{H}^L , one can see that the diagram $r_{-L_1}(\mathcal{H}^L)$ is isotopic to the standard genus-0 diagram of the unknot with one free basepoint and two intersection points \mathbf{x}, \mathbf{y} of attaching curves. Let C^u be the chain complex of $\mathbb{F}[[U_1, U_2]]$ -modules generated by \mathbf{x}, \mathbf{y} , with differential $\partial \mathbf{x} = (U_1 - U_2)\mathbf{y}$. Thus there is a chain homotopy equivalence by counting holomorphic triangles from $A_{\pm\infty, s_2}^-$ to C^u , denoted by $F : A_{\pm\infty, s_2}^- \rightarrow C^u$.

Actually, if we consider the Heegaard diagram by removing z_1 from the Schubert Heegaard diagram, then an area filtration argument shows this chain homotopy equivalence $F : A_{+\infty, s_2}^- \rightarrow C^u$ is in the form of

$$\begin{aligned} F(b_0) &= \mathbf{x} + \text{lower terms}, \\ F(b_{-1}) &= \mathbf{y} + \text{lower terms}, \end{aligned}$$

where the lower terms are referred to the area filtration. In fact, as long as a chain map $G : C^u \rightarrow A_{+\infty, s_2}^-$ is in the form of

$$\begin{aligned} G(\mathbf{x}) &= b_0 + \text{lower terms}, \\ G(\mathbf{y}) &= b_{-1} + \text{lower terms}, \end{aligned}$$

then it is a quasi-isomorphism. This is because $F \circ G : C^u \rightarrow C^u$ is in the form of

$$\begin{aligned} F \circ G(\mathbf{x}) &= \mathbf{x} + \text{lower terms}, \\ F \circ G(\mathbf{y}) &= \mathbf{y} + \text{lower terms}, \end{aligned}$$

which is an isomorphism of groups by Lemma 9.10 in [12]. Thus, adding this condition, we don't have to check the homology of $A_{\pm\infty, s_2}^-$. In order to find an area filtration, we can set every bigon and square to be of the same area 1 on the Schubert Heegaard diagram so that every periodic domain has area 0.

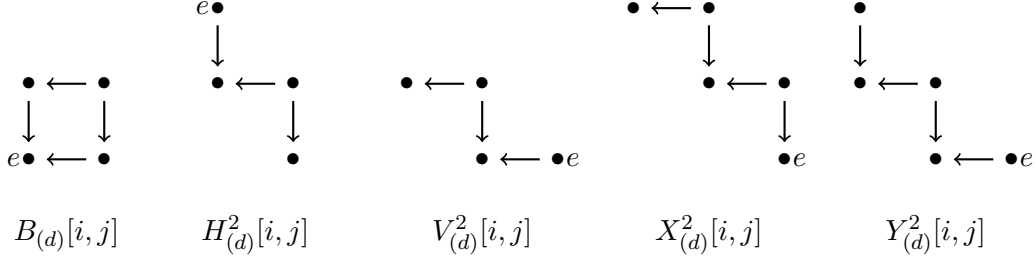


FIGURE 6.1. **Examples of the \mathbb{Z}^2 -filtered chain complexes B, H, V, X, Y .** The labelled dots e in $B_{(d)}[i, j], H_{(d)}^l[i, j], V_{(d)}^l[i, j], X_{(d)}^l[i, j], Y_{(d)}^l[i, j]$ are of grading d and with the filtrations $(i, j), (i, j), (i, j), (i + l, j), (i + l, j)$ respectively.

Third, we plug in all the maps $I_s^{\vec{M}}$ and \tilde{D}^{-L_i} to (5.4). Then we determine the diagonal maps \tilde{F} 's to make those rectangles to be hyperboxes of chain complexes. This is also a problem of solving linear equations without any restrictions.

Finally, we compress all the rectangles and do Λ -twisted gluing of these squares. For computing the homology over polynomial rings over \mathbb{F} , there are several algorithms of polynomial time.

We note that the surgery complex is infinitely generated. Hence, before computing the homology, we need to do truncations for a fixed framing matrix Λ , as described in [5] Section 8.3. The time complexity of doing truncations is a polynomial of $\det(\Lambda)$.

6. EXAMPLES

6.1. The complexes $\widehat{CFL}(L)$ for two-bridge links L . We recall from [14] that for a link L , the filtered chain complex $\widehat{CFL}(L)$ is a chain complex of S^3 with a filtration induced from L . More precisely, fixing a Heegaard diagram \mathcal{H}^L of $L \subset S^3$, we obtain a chain complex of \mathbb{F} -modules $\widehat{CF}(\mathcal{H}^L)$, generated by the intersection points of \mathbb{T}_α and \mathbb{T}_β in the symmetric product. There is an Alexander filtration on $\widehat{CF}(\mathcal{H}^L)$. It is shown that given different Heegaard diagrams of L , \mathcal{H}_1^L and \mathcal{H}_2^L , there is a chain homotopy equivalence from $\widehat{CF}(\mathcal{H}_1^L)$ to $\widehat{CF}(\mathcal{H}_2^L)$, which preserves the Alexander filtration. Thus, the filtered chain homotopy equivalent class of these chain complexes is called the *filtered chain homotopy type* of $\widehat{CFL}(L)$. By abuse of notation, we also let $\widehat{CFL}(L)$ be some filtered complex in this equivalent class. Similarly, we define the filtered chain homotopy type of $CFL^-(L)$, by looking at the Alexander filtered chain complex $CF^-(\mathcal{H}^L)$.

We represent \mathbb{Z}^2 -filtered complexes graphically by dots and arrows on the x - y coordinate plane, with the dots representing generators, the arrows representing differentials, and the coordinates representing filtrations.

Theorem 6.1. (Ozsváth-Szabó, [14], Theorem 12.1) *Suppose $\vec{L} = \vec{L}_1 \cup \vec{L}_2$ is an oriented two-component alternating link. Then the filtered chain homotopy type of $\widehat{CFL}(L)$ is determined by the following data:*

- (1) *the multi-variable Alexander polynomial of L , Δ_L ;*
- (2) *the signature of L , $\sigma(L)$, and the linking number of L , $\text{lk}(L)$;*
- (3) *the filtered chain homotopy type of $\widehat{CFK}(L_1)$ and $\widehat{CFK}(L_2)$.*

In fact, for alternating two-component links, $\widehat{CFL}(L)$ is filtered chain homotopy equivalent to a simplified filtered chain complex $\widehat{CFL}_{\text{OS}}(L)$. The simplified complex is a direct sum of five different types of $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complexes $B_{(d)}[i, j], H_{(d)}^l[i, j], V_{(d)}^l[i, j], X_{(d)}^l[i, j]$ and $Y_{(d)}^l[i, j]$. These basic filtered complexes are described in Section 12.1 of [14]; the filtered complex $B_{(d)}[i, j]$ looks like a box and the others look like zigzags. See Figure 6.1.

Corollary. *If L is an oriented two-bridge link, then the filtered homotopy type of $\widehat{CFL}(L)$ is determined by $\sigma(L)$, $lk(L)$ and the multi-variable Alexander polynomial $\Delta_L(x, y)$. More concretely, let*

$$l = lk(\vec{L}) + \frac{\sigma(\vec{L}) + 1}{2},$$

(1) if $l \geq 0$, let $a = \frac{1-\sigma-lk}{2}$, $b = \frac{-1-\sigma-lk}{2}$, then we have that $\widehat{CFL}(L)$ is filtered chain homotopic to

$$Y_{(0)}^l[a, a] \oplus Y_{(-1)}^{l+1}[b, b] \oplus \bigoplus_k B_{(d_k)}[i_k, j_k],$$

where those d_k, i_k, j_k 's are determined by the Alexander polynomial Δ_L ;

(2) if $l \leq 0$, then we have that the $\widehat{CFL}(L)$ is filtered chain homotopic to

$$X_{(0)}^{|l|}[\frac{lk}{2}, \frac{lk}{2}] \oplus X_{(-1)}^{|l|-1}[\frac{lk}{2}, \frac{lk}{2}] \oplus \bigoplus_k B_{(d_k)}[i_k, j_k],$$

where those d_k, i_k, j_k 's are determined by the Alexander polynomial Δ_L .

Example 6.2. Let Wh denote the Whitehead link. Since $lk(Wh) = 0$, $\sigma(Wh) = -1$, we get $l = -1$. Notice that $lk = 0$ implies that the signature doesn't depend on the orientations of the link. Thus the filtered chain homotopy type of $\widehat{CFL}(Wh)$ is

$$X_{(0)}^1[0, 0] \oplus X_{(-1)}^0[0, 0] \oplus \bigoplus_k B_{(d_k)}[i_k, j_k],$$

where those d_k, i_k, j_k are determined by the Alexander polynomial. If we consider the mirror of Wh , we have $\sigma(\overline{Wh}) = 1$. Similarly, the filtered chain homotopy type of $\widehat{CFL}(\overline{Wh})$ is

$$Y_{(0)}^1[0, 0] \oplus Y_{(-1)}^0[0, 0] \oplus \bigoplus_k B_{(d'_k)}[i'_k, j'_k].$$

In the following diagram, $\widehat{CFL}(Wh)$ and $\widehat{CFL}(\overline{Wh})$ are illustrated, where each dot represents a generator and each arrow represents a differential.

$$(6.1) \quad \begin{array}{ccc} \sigma(\overline{Wh}) = 1 : & \bullet \leftarrow \bullet \bullet \leftarrow \bullet & \sigma(Wh) = -1 : \bullet \leftarrow \bullet \bullet \leftarrow \bullet \\ \downarrow & \downarrow \downarrow \downarrow \downarrow & \downarrow \\ \bullet \leftarrow \bullet \bullet \leftarrow \bullet & & \bullet \leftarrow \bullet \bullet \leftarrow \bullet \\ \downarrow & \downarrow \downarrow \downarrow \downarrow & \downarrow \\ \bullet \leftarrow \bullet \bullet \leftarrow \bullet, & & \bullet \leftarrow \bullet \bullet \leftarrow \bullet. \end{array}$$

We find that it is easier to work with Wh with $\sigma = -1$, since all the $A_s^-(Wh)$ have homology $\mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$, so that we can apply the rigidity results. (One can compare this to the case of the left-handed trefoil knot versus the right-handed trefoil knot.)

Remark 6.3. The way of decomposing the complex $\widehat{CFL}(\vec{L})$ into direct sums of B, H, V, X, Y is not canonical. We can do some base-changes to change the above direct sum decomposition of \widehat{CFL} such that the patterns of the arrows don't change.

6.2. The filtered homotopy type of $CFL^-(L)$ for some two-bridge links. Given a two-bridge link L , we can use the Schubert Heegaard diagram to combinatorially find the filtered complex $CFL^-(L)$. However, this description is too cumbersome. Instead, here we use algebraic arguments to determine the filtered homotopy type of $CFL^-(L)$ in some special examples.

Consider the Schubert Heegaard diagram \mathcal{H} for L and the \mathbb{Z}^2 -filtered chain complex $\widehat{CF}(\mathcal{H})$. Then there is a filtered chain homotopy equivalence $F : \widehat{CF}(\mathcal{H}) \rightarrow \widehat{CFL}_{\text{OS}}(L)$. Thus, F induces an isomorphism on the homology of their associated graded, i.e. the link Floer homology. In fact, the homology of their associated graded are just the chain groups themselves, because $\widehat{CF}(\mathcal{H})$ and

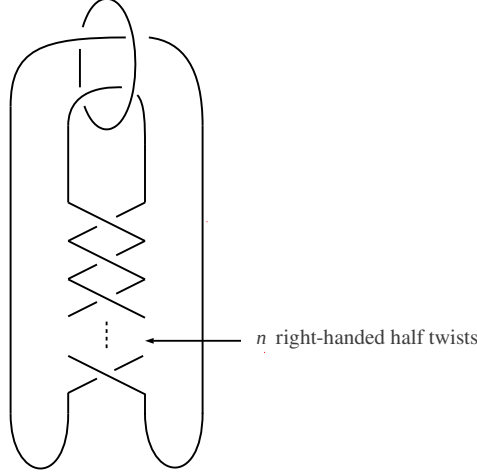


FIGURE 6.2. **The two-bridge links $b(4n + 4, 2n + 3)$.** If $n = 2k - 1$, then the linking number is 0; if $n = 2k$, then the linking number is 2.

$\widehat{CFL}_{OS}(L)$ are both thin (with no differentials in their associated graded). So F is an isomorphism. In other words, we can change the basis of $\widehat{CF}(\mathcal{H})$ preserving the filtration, such that the arrows are pruned as in the Ozsváth-Szabó simplified pattern. We use this new basis to consider $CF^-(\mathcal{H})$. Since every bigon in \mathcal{H} contains a basepoint, those ∂_{U_1, U_2} arrows in $CF^-(\mathcal{H})$ are either upward or rightward of length 1. This property is repeatedly used later.

In this section, we show that for the two-bridge links $b(4n, 2n + 1)$, the filtered homotopy type of $CFL^-(L)$ is determined by $\widehat{CFL}(L)$. Since $\widehat{CFL}(L)$ can be decomposed as direct sums of B, X, Y 's, our goal is to show that $CFL^-(L)$ can be viewed as a square of chain complexes of these B, X, Y 's.

Using continuous fractions, we can get the 4-plat presentations of $b(4n, 2n + 1)$, thus providing the diagram in Figure 6.2. In addition, there is a convention issue of signs of the signature. We adopt the convention compatible with Theorem 6.1 in [14], so that $\sigma(b(8k, 4k + 1)) = -1$.

Proposition 6.4. *For the two-bridge link $L = b(8k, 4k + 1)$, the filtered homotopy type of $CFL^-(L)$ is determined by the Alexander polynomial, signature and linking number of L , or equivalently by $\widehat{CFL}(L)$. Precisely, we have $CFL^-(L) = CFL^-(Wh) \oplus \bigoplus_{i=1}^{k-1} (N, \partial^-)$, where $CFL^-(Wh)$ and (N, ∂^-) are described in Figure 6.3.*

Proof. By Theorem 6.1, the Ozsváth-Szabó simplified complex $\widehat{CFL}_{OS}(L)$ can be computed in terms of $\Delta_L(x, y) = k\Delta_{Wh}(x, y) = k\frac{(x-1)(y-1)}{\sqrt{xy}}$, $lk = 0$, and $\sigma(L) = \pm 1$. In fact, $\sigma(b(8k, 4k + 1)) = -1$, and $\sigma(b(8k, 4k - 1)) = 1$. Since there is a convention issue of signs of the signature, let us consider the case where $\sigma = -1$. Then, we compute that

$$\widehat{CFL}_{OS}(L) = A \oplus B \oplus C \oplus D = \bigoplus_{i=1}^k A^{(i)} \oplus \bigoplus_{i=1}^k B^{(i)} \oplus \bigoplus_{i=1}^k C^{(i)} \oplus \bigoplus_{i=1}^k D^{(i)}.$$

See Figure 6.4 for the filtered homotopy type of $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$, where we denote the generators in $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$ by $a_j^{(i)}, b_j^{(i)}, c_j^{(i)}, d_j^{(i)}$, $j = 1, 2, 3, 4$, respectively.

Given $\widehat{CFL}_{OS}(L)$, let us investigate the possibilities for $CFL^-(L)$. The differential ∂^- in $CFL^-(L)$ decomposes into

$$\partial^- = \hat{\partial} + \partial_{U_1, U_2} = \partial_{A_1} + \partial_{A_2} + \partial_{U_1} + \partial_{U_2},$$

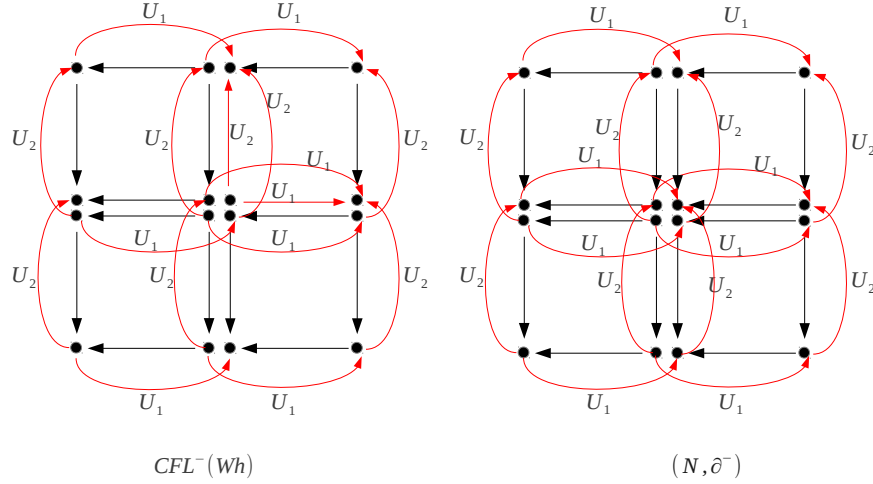


FIGURE 6.3. **The filtered complex $CFL^-(Wh)$ and (N, ∂^-) .** The horizontal red arrows and vertical red arrows have U_1 and U_2 coefficients respectively.

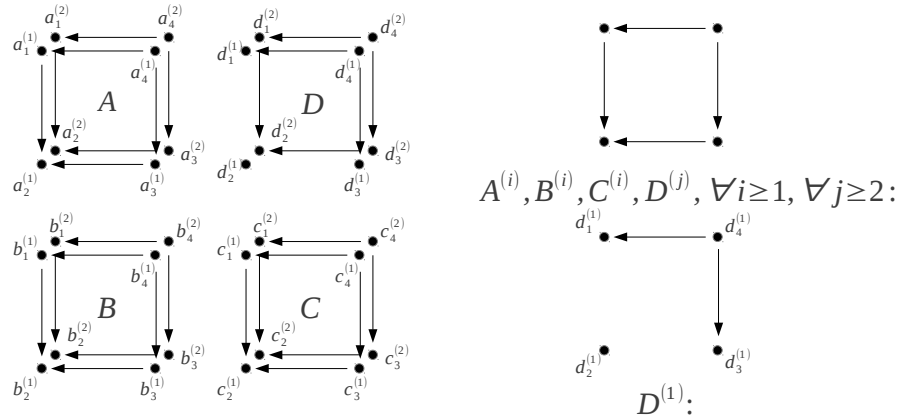


FIGURE 6.4. $\widehat{CFL}_{OS}(b(8k, 4k \pm 1))$. On the left side, the figure illustrates the Alexander grading of A, B, C, D summands, where $k = 2$. On the right side, it indicates the filtered homotopy types of $(A^{(i)}, \hat{\partial}), (B^{(i)}, \hat{\partial}), (C^{(i)}, \hat{\partial}), (D^{(i)}, \hat{\partial})$, which all have the filtered homotopy types as boxes, except for $(D^{(1)}, \hat{\partial})$.

where $\partial_{U_1, U_2}(x) = \partial_{U_1}(x) + \partial_{U_2}(x)$ consists of the components in $\partial^-(x)$ with coefficients of U_1, U_2 powers, and $\hat{\partial}(x) = \partial_{A_1}(x) + \partial_{A_2}(x)$ is decomposed by the Alexander filtration. As stated before, here ∂_{U_i} has the form of $\partial_{U_i}(x) = U_i y$ for $i = 1, 2$, i.e. the ∂_{U_i} -arrows are all of length 1. A close examination of U_1, U_2 powers and the Alexander filtrations in the coefficients of the following identity provides that

$$0 = (\partial^-)^2 = (\hat{\partial} + \partial_{U_1, U_2})^2 = (\partial_{A_1} + \partial_{A_2} + \partial_{U_1} + \partial_{U_2})^2 \implies [\hat{\partial}, \partial_{U_1, U_2}] = 0, \partial_{U_1, U_2}^2 = \hat{\partial}^2 = 0 \implies [\partial_{A_1}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_2}] = [\partial_{U_1}, \partial_{U_2}] = 0, \partial_{A_1}^2 = \partial_{U_1}^2 = \partial_{A_2}^2 = \partial_{U_2}^2 = 0.$$

where $[f, g] = fg + gf$.

At this point, we first consider the Whitehead link. The $\widehat{CFL}(Wh)$ is shown by the right term in Equation (6.1), and the bullets are labeled as in Figure 6.4. By looking at the vertical arrows

only, the equations $\partial_{U_2}^2 = \partial_{A_2}^2 = [\partial_{A_2}, \partial_{U_2}] = 0$ give rise to the two possibilities of the rightmost column as follows, according to whether $\partial_{U_2}(c_4^{(1)})$ is 0 or not.



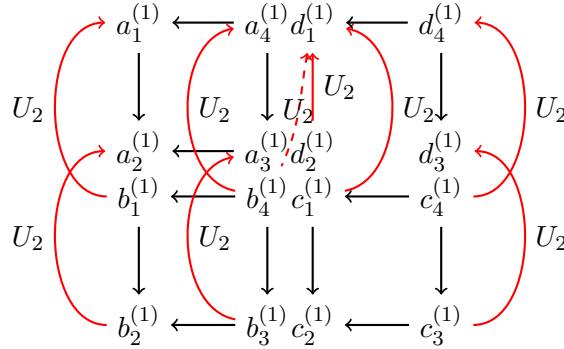
Consider the Heegaard diagram of L_1 obtained from \mathcal{H} by deleting w_2 , i.e. the reduction $r_{-L_2}(\mathcal{H})$. The differentials in $\widehat{CF}(r_{-L_2}(\mathcal{H}))$ count the bigons without basepoints w_1, z_1, z_2 on \mathcal{H} , which are the same as bigons with basepoint w_2 . Thus, the complex $\widehat{CF}(r_{-L_2}(\mathcal{H}))$ can be obtained by ignoring the arrows ∂_{A_2} and setting $U_2 = 1$. So the vertical homology of $CFL^-(L)$ using only the ∂_{U_2} arrows is the knot Floer homology of the unknot L_1 , $\mathbb{F} \oplus \mathbb{F}$, supported in the filtration $A_1 = \tau(L_1) + \frac{\text{lk}(L)}{2} = 0$. Thus, the right-hand side in the above diagram is ruled out. A similar argument applies to the leftmost column. In sum,

$$\partial_{U_2}(b_1^{(1)}) = U_2 a_1^{(1)}, \quad \partial_{U_2}(b_2^{(1)}) = U_2 a_2^{(1)}, \quad \partial_{U_2}(c_4^{(1)}) = U_2 d_4^{(1)}, \quad \partial_{U_2}(c_3^{(1)}) = U_2 d_3^{(1)}.$$

Together with $\partial_{A_1} \partial_{U_2} = \partial_{U_2} \partial_{A_1}$, we get

$$\partial_{A_1} \partial_{U_2}(b_4^{(1)}) = \partial_{U_2} \partial_{A_1}(b_4^{(1)}) = \partial_{U_2} b_1^{(1)} = U_2 a_1^{(1)} \implies \partial_{U_2}(b_4^{(1)}) = U_2 a_4^{(1)} \text{ or } U_2(a_4^{(1)} + d_1^{(1)}).$$

Thus, $\partial_{A_2} \partial_{U_2} = \partial_{U_2} \partial_{A_2}$ implies that $\partial_{U_2}(b_3^{(1)}) = a_3^{(1)}$ and $\partial_{U_2}(c_1^{(1)}) = d_1^{(1)}$, $\partial_{U_2}(c_2^{(1)}) = 0$.



Next, $\partial_{U_2} \partial_{A_2} = \partial_{A_2} \partial_{U_2}$ implies that $\partial_{U_2}(d_2^{(1)}) \in U_2 \cdot D$. Consider the complex $CFL^-(L) \otimes (\mathbb{F}[[U_1, U_2]]/U_1) = CFL^-(L)/(U_1 \cdot CFL^-(L))$, i.e. setting $U_1 = 0$. The homology can be computed from the exact sequence of the homologies, which is $\mathbb{F}[[U_2]]/U_2$ as an $\mathbb{F}[[U_2]]$ -module. Taking the vertical homology of this complex with respect to ∂_{A_2} leaves only $d_2^{(1)}$ and $d_1^{(1)}$. It follows that $\partial_{U_2}(d_2^{(1)}) = U_2 d_1^{(1)}$. Thus, we recover all the U_2 -arrows. See the above figure, where the dashed arrow is undetermined. Similarly, we can get all the U_1 -arrows. By changing basis, $\tilde{a}_4^{(1)} = a_4^{(1)} + d_1^{(1)}$ and $\tilde{c}_4^{(1)} = c_4^{(1)} + d_3^{(1)}$, we can get rid of the dashed arrows, which gives the picture of $CFL^-(Wh)$ in Figure 6.3.

When $k > 1$, we follow the same line of argument, together with doing more changes of basis to prune the arrows. First, consider the rightmost column, i.e. $R = \text{Span}_{\mathbb{F}[[U_2]]}\{d_3^{(i)}, d_4^{(i)}, c_3^{(i)}, c_4^{(i)}\}_{i=1}^k$ with the differentials $\partial_{A_2} + \partial_{U_2}$. Assume $\partial_{U_2}(c_4^{(i)}) = U_2 \cdot \sum_{m=1}^k \lambda_{i,m} d_4^{(m)}$. Then $\partial_{U_2} \partial_{A_2} = \partial_{A_2} \partial_{U_2}$

implies that $\partial_{U_2}(c_3^{(i)}) = U_2 \cdot \sum_{m=1}^k \lambda_{i,m} d_3^{(m)}$. So the matrix $D = (\lambda_{i,m})$ represents the differential ∂_{U_2} in the upside down vertical complex $(R \otimes \mathbb{F}[[U_2]]/(U_2 - 1), \partial_{U_2})$. Since its homology is 0, the matrix D is invertible. In other words, the ∂_{U_2} -arrows form an isomorphism from $\text{Span}_{\mathbb{F}}\{c_3^{(i)}, c_4^{(i)}\}_{i=1}^k$ to $\text{Span}_{\mathbb{F}}\{d_3^{(i)}, d_4^{(i)}\}_{i=1}^k$. Thus, we can find a new basis of C , namely $\{\tilde{c}_1^{(i)}, \tilde{c}_2^{(i)}, \tilde{c}_3^{(i)}, \tilde{c}_4^{(i)}\}_{i=1}^k$, such that

$$\partial_{U_2}(\tilde{c}_3^{(i)}) = U_2 \cdot d_3^{(i)}, \quad \partial_{U_2}(\tilde{c}_4^{(i)}) = U_2 \cdot d_4^{(i)}, \forall 1 \leq i \leq k,$$

while the pattern of the $\hat{\partial}$ is preserved. In addition, $[\partial_{U_2}, \partial_{A_1}] = 0$ implies that

$$\begin{aligned} \partial_{U_2}(\tilde{c}_1^{(i)}) &= U_2 \cdot d_1^{(i)}, \quad \partial_{U_2}(\tilde{c}_2^{(i)}) = U_2 \cdot d_2^{(i)}, \forall 2 \leq i \leq k, \\ \partial_{U_2}(\tilde{c}_1^{(1)}) &= U_2 \cdot d_1^{(1)}, \quad \partial_{U_2}(\tilde{c}_2^{(1)}) = 0. \end{aligned}$$

From the fact that the vertical homology of $CFL^-(L)$ with respect to the differential $\partial_{A_2} + \partial_{U_2}$ is $\mathbb{F}[[U_2]]/U_2$, it follows that $\partial_{U_2}(d_2) = U_2 \cdot d_1$.

We may as well keep using the notations $c_j^{(i)}$ for the new basis. Applying similar arguments for the leftmost column with respect to vertical arrows, we can change the basis of A without changing the pattern of $\hat{\partial}$, such that

$$\partial_{U_2}(b_j^{(i)}) = U_2 a_j^{(i)}, \forall j = 1, 2, \forall i = 1, \dots, k.$$

Then $\partial_{A_1} \partial_{U_2} = \partial_{U_2} \partial_{A_1}$ implies

$$\partial_{U_2}(b_4^{(i)}) = U_2 a_4^{(i)} + \sum_{m=1}^k \varepsilon_{i,m} U_2 d_1^{(m)}, \forall i = 1, \dots, k, \varepsilon_{i,m} \in \mathbb{F}.$$

Thus, $\partial_{U_2}(b_3^{(i)}) = U_2 a_3^{(i)} + \sum_{m=2}^k \varepsilon_{i,m} U_2 d_2^{(m)}, \forall i = 1, \dots, k$. Do base-changes:

$$\tilde{a}_4^{(i)} = a_4^{(i)} + \sum_{m=1}^k \varepsilon_{i,m} d_1^{(m)}, \quad \tilde{a}_3^{(i)} = a_3^{(i)} + \sum_{m=2}^k \varepsilon_{i,m} d_2^{(m)}.$$

We can preserve the pattern of $\hat{\partial}$, such that under the new basis (where we keep using the notations $a_j^{(i)}$) all the vertical arrows are pruned as $\partial_{U_2}(b_j^{(i)}) = U_2 a_j^{(i)}, \forall j = 1, 2, 3, 4, \forall i = 1, \dots, k$. Similarly, by changing the bases of A and B simultaneously, we can prune the horizontal arrows in the top row, while preserving the pattern of $\hat{\partial}, \partial_{U_2}$ on A and B , such that $\partial_{U_1}(b_j^{(i)}) = U_1 c_j^{(i)}, \forall j = 1, 4, \forall i = 1, \dots, k$. Then $\partial_{A_2} \partial_{U_1} = \partial_{U_1} \partial_{A_2}$ implies that $\partial_{U_1}(b_2^{(i)}), \partial_{U_1}(b_3^{(i)})$ are determined.

Similarly, all the horizontal arrows from B can be pruned by changing the basis of C . Suppose

$$\partial_{U_1}(b_4^{(i)}) = U_1 \left(\sum_{m=1}^k \lambda_{i,m} c_4^{(m)} + \sum_{m=1}^k \mu_{i,m} d_3^{(m)} \right).$$

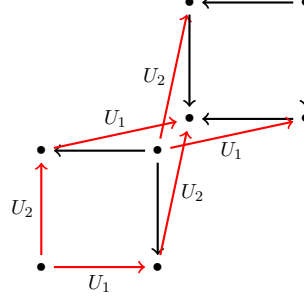
Then $\partial_{U_1} \partial_{U_2}(b_4^{(i)}) = \partial_{U_2} \partial_{U_1}(b_4^{(i)})$ implies that $U_1 U_2 (\sum_{m=1}^k \lambda_{i,m} d_4^{(m)}) = U_1 U_2 d_4^{(i)}$. Thus, $\lambda_{i,i} = 1, \lambda_{i,m} = 0, \forall i = 1, \dots, k, \forall m \neq i$. Thus, $\partial_{U_1}(b_4^{(i)}) = U_1 (c_4^{(i)} + \sum_{m=1}^k \mu_{i,m} d_3^{(m)})$. Do base-changes

$$\tilde{c}_4^{(i)} = c_4^{(i)} + \sum_{m=1}^k \mu_{i,m} d_3^{(m)}, \quad \tilde{c}_1^{(i)} = c_1^{(i)} + \sum_{m=2}^k \mu_{i,m} d_2^{(m)}, \forall i = 1, \dots, k.$$

Under the new basis (where we keep using the notations $c_j^{(i)}$), the patterns of all the $\hat{\partial}$ and ∂_{U_2} arrows are preserved, while $\partial_{U_1}(b_4^{(i)}) = c_4^{(i)}$. Moreover, $[\partial_{A_1}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_1}] = 0$ implies that $\partial_{U_1} b_j^{(i)} = U_1 c_j^{(i)}, \forall j = 1, 2, 3, 4, \forall i = 1, \dots, k$. Finally, all the arrows are as in Figure 6.3. Thus, $CFL^-(L)$ can be viewed as a square of chain complexes of A, B, C, D . \square

Similar arguments apply to the case of $L = b(8k + 4, 4k + 3)$.

Proposition 6.5. *For the two-bridge link $L = b(8k + 4, 4k + 3)$, the filtered homotopy type of $CFL^-(L)$ is determined by the filtered homotopy type of $\widehat{CFL}(L)$ (and hence by the Alexander polynomial, signature and linking number). Furthermore, $CFL^-(L) = CFL^-(T(4, 2)) \oplus \bigoplus_{i=1}^{k-1} (N, \partial^-)$, where (N, ∂^-) is as in Figure 6.3 and $CFL^-(T(4, 2))$ is as follows.*



6.3. Computations of surgeries on $b(8k, 4k + 1)$. In this section, we compute the homology of surgeries on the two-bridge link $b(8k, 4k + 1)$ and their d -invariants explicitly. Here, we make a convention of the d -invariants of \mathbf{HF}^- different from [13]. We require that $d(\mathbf{HF}^-(S^3)) = 0$. Thus, the d -invariants computed here are the same as the d -invariants for HF^+ .

We will first compute for the Whitehead link, following three steps: computations of $A_s^-(Wh)$, computations of the inclusion maps $I_s^{\vec{M}}$, and the computations of the homology of the surgeries on Wh .

Lemma 6.6. $H_*(A_s^-(Wh)) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2) = \mathbb{F}[[U]]$ for all $s \in \mathbb{H}(Wh) = \mathbb{Z}^2$.

Proof. By Proposition 6.4, we can decompose $A_{+\infty, +\infty}^-(Wh)$ into a square of chain complexes, i.e.

$$\begin{array}{ccc} A & \rightarrow & D \\ \uparrow & & \uparrow \\ B & \rightarrow & C, \end{array}$$

where the summands A, B, C, D are described in Proposition 6.4. Since $A_{+\infty, +\infty}^-$ can be viewed as a mapping cone from $A \oplus B \oplus C$ to D , we get a short exact sequence of chain complexes

$$0 \rightarrow D \xrightarrow{i} A_{+\infty, +\infty}^- \rightarrow A \oplus B \oplus C \rightarrow 0.$$

From the fact that $H_*(A \oplus B \oplus C) = 0$, it follows that i is a quasi-isomorphism. Hence $H_*(A_{+\infty, +\infty}^-) = \mathbb{F}[[U]]$ and $[d_1] = [d_3] = 1 \in H_*(A_{+\infty, +\infty}^-)$.

All the other complexes $A_{s_1, s_2}^-(Wh)$ can be actually obtained by taking various reflections on $A_{+\infty, +\infty}^-(Wh)$. Note that Equation (2.2) implies that the differentials in A_s^- are only changed by U_1, U_2 powers from $A_{+\infty, +\infty}^-$. In order to read off the correct powers of U_1, U_2 -coefficients, we can change the \mathbb{Z}^2 -filtration of A_s^- , such that the upward and rightward arrows in A_s^- are with U_1 and U_2 coefficients respectively. For instance, when $s_1 > 0, s_2 = 0$, we can flip the summands A and D about the A_1 -axis to obtain the complex $A_{s_1, 0}^-$. For convenience, we denote the vertical reflections of A, B, C, D by $\overline{A}, \overline{B}, \overline{C}, \overline{D}$. Thus, the complex $A_{s_1, 0}^-$ is still a square of chain complexes as follows:

$$\begin{array}{ccc} \overline{A} & \rightarrow & \overline{D} \\ \uparrow & & \uparrow \\ B & \rightarrow & C. \end{array}$$

Thus, the fact that \overline{A}, B, C are acyclic implies that $H_*(A_{s_1, 0}^-) = H_*(\overline{D}) = \mathbb{F}[[U]]$. Similarly, we denote the horizontal reflections of A, B, C, D by $|A, |B, |C, |D$ respectively. Thus, the complex

A_{0,s_1}^- with $s_1 > 0$ is the following square of chain complexes

$$\begin{array}{ccc} A & \rightarrow & |D \\ \uparrow & & \uparrow \\ B & \rightarrow & |C. \end{array}$$

Following the same line, we list all the filtered homotopy types of A_s^- 's together with some generators of their homologies as follows. Since $\max A_1 = \max A_2 = 1$, $\min A_1 = \min A_2 = -1$, the notation $+\infty$ means a positive integer s , while $-\infty$ means a negative integer s .

$A_{-\infty,+\infty}^- =$ $ A \rightarrow D$ $\uparrow \quad \uparrow$ $ B \rightarrow C,$ $[d_1] = 1 \in H_*(A_{-\infty,+\infty}^-);$	$A_{0,+\infty}^- =$ $A \rightarrow D$ $\uparrow \quad \uparrow$ $B \rightarrow C,$ $[d_1] = 1 \in H_*(A_{0,+\infty}^-);$	$A_{+\infty,+\infty}^- =$ $A \rightarrow D$ $\uparrow \quad \uparrow$ $B \rightarrow C,$ $[d_1] = [d_3] = 1 \in H_*(A_{+\infty,+\infty}^-);$
$A_{-\infty,0}^- =$ $ \bar{A} \rightarrow \bar{D}$ $\uparrow \quad \uparrow$ $ B \rightarrow C,$ $[a_2] = 1 \in H_*(A_{-\infty,0}^-);$	$A_{0,0}^- =$ $\bar{A} \rightarrow \bar{D}$ $\uparrow \quad \uparrow$ $B \rightarrow C,$ $[d_1] = [d_3] = 1 \in H_*(A_{0,0}^-);$	$A_{+\infty,0}^- =$ $\bar{A} \rightarrow \bar{D}$ $\uparrow \quad \uparrow$ $B \rightarrow C,$ $[d_3] = 1 \in H_*(A_{+\infty,0}^-);$
$A_{-\infty,-\infty}^- =$ $ \bar{A} \rightarrow \bar{D}$ $\uparrow \quad \uparrow$ $ \bar{B} \rightarrow \bar{C},$ $[a_2] = [c_2] = 1 \in H_*(A_{-\infty,-\infty}^-);$	$A_{0,-\infty}^- =$ $\bar{A} \rightarrow \bar{D}$ $\uparrow \quad \uparrow$ $\bar{B} \rightarrow \bar{C},$ $[c_2] = 1 \in H_*(A_{0,-\infty}^-);$	$A_{+\infty,-\infty}^- =$ $\bar{A} \rightarrow \bar{D}$ $\uparrow \quad \uparrow$ $\bar{B} \rightarrow \bar{C},$ $[d_3] = [c_2] = 1 \in H_*(A_{+\infty,-\infty}^-).$

TABLE 6.1. $A_s^-(Wh)$ and generators of their homology.

Note that $A, \bar{A}, |A, B, \bar{B}, |B, C, \bar{C}, |C$ are all acyclic, and $D, \bar{D}, |D, |\bar{D}$ all have the same homology $\mathbb{F}[[U]]$. We can use the same argument for $A_{+\infty,+\infty}^-$ to show that $A_{-\infty,+\infty}^-$, $A_{0,+\infty}^-$, $A_{+\infty,+\infty}^-$, $A_{-\infty,0}^-$, $A_{+\infty,0}^-$, and $A_{+\infty,-\infty}^-$ all have the same homology $\mathbb{F}[[U]]$. For those other A_s^- , we can use the conjugation symmetry, that is, $H_*(A_s^-(L)) = H_*(A_{-s}^-(L))$, $\forall s \in \mathbb{H}(L)$, $\forall L$. This is because A_s^- 's are quasi-isomorphic to the Floer complexes of large surgeries on L .

Now we explain the generators of their homologies in Table 6.1. The chain complex $A_{0,-\infty}^-$ can be viewed as a mapping cone of a chain map from $\text{cone}(\bar{B} \rightarrow \bar{A})$ to $\text{cone}(|\bar{C} \rightarrow |\bar{D})$. Because $\text{cone}(\bar{B} \rightarrow \bar{A})$ is acyclic, the generator of $H_*(\text{cone}(|\bar{C} \rightarrow |\bar{D}))$ is also a generator of $H_*(A_{0,-\infty}^-)$. Since $H_*(|\bar{C}) = \mathbb{F}[[U]]/U$, $H_*(|\bar{D}) = \mathbb{F}[[U]]$, we derive a short exact sequence

$$0 \rightarrow \mathbb{F}[[U]] \rightarrow \mathbb{F}[[U]] \rightarrow \mathbb{F}[[U]]/U \rightarrow 0$$

from the long exact sequence of the homologies $\cdots \rightarrow H_*(|\bar{D}) \rightarrow H_*(\text{cone}(|\bar{C} \rightarrow |\bar{D})) \rightarrow H_*(|\bar{C}) \rightarrow \cdots$. Because $[c_2] = 1 \in H_*(|\bar{C}) = \mathbb{F}[[U]]/U$ and $[c_2] \in \text{cone}(|\bar{C} \rightarrow |\bar{D})$ is mapped to $[c_2] \in H_*(|\bar{C})$, the above short exact sequence implies that $[c_2] = 1 \in H_*(\text{cone}(|\bar{C} \rightarrow |\bar{D}))$, and thus $[c_2] = 1 \in H_*(A_{0,-\infty}^-)$.

Similar arguments show that $[a_2] = [c_2] = 1 \in H_*(A_{-\infty,-\infty}^-)$ and $[d_3] = 1 \in H_*(A_{+\infty,-\infty}^-)$. Moreover, in the complex $A_{+\infty,-\infty}^-$, the equations $\partial^- c_1 = U_2 c_2 + d_1$, $\partial^- d_2 = d_1 + U_1 d_3$ imply that $[c_2] = [d_3] = 1 \in H_*(A_{+\infty,-\infty}^-)$. \square

Taking the grading into account, we adopt the formula of the $\mathbb{Z}/\mathfrak{d}(\mathfrak{u})\mathbb{Z}$ -grading defined on the surgery complex for a Spin^c structures $\mathfrak{u} \in \mathbb{H}(L)/H(L, \Lambda)$ in [5] Section 7.4,

$$(6.2) \quad \mu(s, \mathbf{x}) = \mu_s^M(\mathbf{x}) + \nu(s) - \|M\|, \mathbf{x} \in \mathfrak{A}^-(\mathcal{H}^{L-\vec{M}}, \psi^{\vec{M}}(s)),$$

where $s \in \mathfrak{u}$ and $\mu_s^M = \mu_{\psi^{\vec{M}}(s)}$ is a natural \mathbb{Z} -grading defined on $\mathfrak{A}^-(L - \vec{M}, \psi^{\vec{M}}(s))$. In the torsion case, the quadratic function ν can be chosen as 0. The natural \mathbb{Z} -grading $\mu_s^\emptyset = \mu_{s_1, s_2}$ on each A_{s_1, s_2}^- is given by

$$\mu_{s_1, s_2}(\mathbf{x}) = M(\mathbf{x}) - 2 \sum_{i=1}^2 \max\{A_i(\mathbf{x}) - s_i, 0\},$$

where $M(\mathbf{x})$ is the Maslov grading. When we use the Schubert Heegaard diagram, $A_1(\mathbf{x}) + A_2(\mathbf{x}) - M(\mathbf{x})$ is constant. Thus, up to a shift of a constant number, we can take $M(\mathbf{x}) = A_1(\mathbf{x}) + A_2(\mathbf{x})$ for $\forall \mathbf{x} \in A_{s_1, s_2}^-$. In the primitive system we identify $\mathfrak{A}^-(L - \vec{M}, \psi^{\vec{M}}(s))$ with some $A_{s'_1, s'_2}^-$ (where s'_1, s'_2 can evaluate $+\infty$), so the grading μ_s^M is actually μ_s . We define some rules of ∞ as follows:

$$0 \cdot (+\infty) = +\infty; s + (+\infty) = +\infty, \forall s \in \mathbb{R}; s + (-\infty) = -\infty, \forall s \in \mathbb{R}; (\pm 1) \cdot (+\infty) = \pm \infty.$$

Recall the notations in Example 4.13. The complexes $C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)} = A_{s_1 + \varepsilon_1 \cdot \infty, s_2 + \varepsilon_2 \cdot \infty}^-$, $\varepsilon_i \in \{0, 1\}$ are setting at the position $(\varepsilon_1, \varepsilon_2)$ in the square and with the index (s_1, s_2) in the product complex $C^{(\varepsilon_1, \varepsilon_2)} = \prod_{s_1, s_2} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}$.

We define the grading $\mu_{s_1, s_2}^{\varepsilon_1, \varepsilon_2}$ on the complex $C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}$ by the formula:

$$\mu_{s_1, s_2}^{\varepsilon_1, \varepsilon_2}(\mathbf{x}) = M(\mathbf{x}) - 2 \sum_{i=1}^2 \max\{A_i(\mathbf{x}) - s_i - \varepsilon_i(+\infty), 0\} - \varepsilon_1 - \varepsilon_2.$$

Here $\mu_{s_1, s_2}^{\varepsilon_1, \varepsilon_2}$ plays the role as μ in Equation (6.2).

Let W be the four-manifold cobordism corresponding to the surgery from S^3 to $S_\Lambda^3(L)$. In [5], it is shown that the cobordism map $F_{W, s}^-$ corresponds to the inclusion $\iota : \mathfrak{A}^-(\mathcal{H}^\emptyset) \rightarrow \mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{\vec{L}}(s)) \subset C^-(\mathcal{H}, \Lambda)$, $\vec{L} = +L_1 \cup +L_2$. So we need to shift the grading such that ι is of the degree $\deg(F_{W, s}^-) = \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}$. In our case, the complex $\mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{\vec{L}}(s)) = C_{(s_1, s_2)}^{(1, 1)} = A_{+\infty, +\infty}^-(Wh)$ has a generator $[d_1] = 1 \in H_*(A_{+\infty, +\infty}^-)$ of Alexander grading $A(d_1) = (0, 1)$. Finally, the grading formula turns out to be

$$(6.3) \quad \mu_{s_1, s_2}^{\varepsilon_1, \varepsilon_2}(\mathbf{x}) = A_1(\mathbf{x}) + A_2(\mathbf{x}) - 2 \sum_{i=1}^2 \max\{A_i(\mathbf{x}) - s_i - \varepsilon_i(+\infty), 0\} - \varepsilon_1 - \varepsilon_2 + \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4} + 1,$$

where $c_1(s) = [2s] - \Lambda_1 - \Lambda_2 = (2s_1 - p_1, 2s_2 - p_2) \in \mathbb{Z}^2/\Lambda$. In the perturbed surgery complex, since all the perturbed maps have the same degrees as the original, we can compute the gradings still using Equation (6.3).

By Proposition 5.2 and Lemma 6.6, up to chain homotopy, all the edge maps $\Phi_s^{\pm L_i}$ are classified by their actions on the homologies. The actions of $\Phi_s^{\pm L_i}$ on homologies are determined by the corresponding inclusion maps $I_s^{\pm L_i}$. We denote the induced maps on homologies by $(I_s^{\pm L_i})_* : \mathbb{F}[[U]] \rightarrow \mathbb{F}[[U]]$.

Lemma 6.7. *Regarding the inclusion maps, we have the following results for $I_s^{\pm L_1}$, where $s = (s_1, s_2)$.*

- If $s_1 > 0$, then $(I_s^{+L_1})_* = id$.
- If $s_1 = 0, s_2 \neq 0$, then $(I_s^{+L_1})_* = id$.
- If $s_1 = s_2 = 0$, then $(I_s^{+L_1})_* = U \cdot id$.

- If $s_1 < 0$, then $(I_s^{+L_1})_* = U^{-s_1} \cdot id$.
- If $s_1 > 0$, then $(I_s^{-L_1})_* = U^{s_1} \cdot id$.
- If $s_1 = 0, s_2 \neq 0$, then $(I_s^{-L_1})_* = id$.
- If $s_1 = s_2 = 0$, then $(I_s^{-L_1})_* = U \cdot id$.
- If $s_1 < 0$, then $(I_s^{-L_1})_* = id$.

Proof. In fact, when $s_1 > 0$, by definition $I_{s_1, s_2}^{+L_1} = id$. When $s_1 = 0, s_2 > 0$, we have $I_{0, s_2}^{+L_1}(d_1) = d_1$. Therefore by Table 6.1, the inclusion map $I_{0, s_2}^{+L_1}$ acts on the homology as $id : \mathbb{F}[[U]] \rightarrow \mathbb{F}[[U]]$. When $s_1 = 0, s_2 < 0$, we have $I_{0, s_2}^{+L_1}(c_2) = c_2$. Therefore by Table 6.1, the inclusion map $I_{0, s_2}^{+L_1}$ acts on homology as the identity. When $s_1 = s_2 = 0$, we have $I_{0, 0}^{+L_1}(d_3) = U_1 \cdot d_3$. Therefore by Table 6.1, $I_{0, 0}^{+L_1}$ acts on homology as $U \cdot id$. When $s_1 < 0, s_2 > 0$, we have $I_{s_1, s_2}^{+L_1}(d_1) = U_1^{-s_1} \cdot d_1$. Thus, $(I_s^{+L_1})_* = U^{-s_1} \cdot id$. When $s_1 < 0, s_2 \leq 0$, we have $I_{s_1, s_2}^{+L_1}(a_2) = U_1^{-1-s_1} \cdot a_2$. In the complex $A_{+\infty, s_2}^-, s_2 \leq 0$, the equation $\partial^- a_3 = a_2 + U_1 d_3$ implies that $[a_2] = U_1[d_3] \in H_*(A_{+\infty, s_2}^-)$. Thus, $[a_2] = U \in \mathbb{F}[[U]] = H_*(A_{+\infty, s_2}^-)$. Therefore, it follows that $(I_{s_1, s_2}^{+L_1})_* = U^{-s_1} \cdot id$, when $s_1 < 0, s_2 \leq 0$. In the same way, we get the following results for $I_s^{-L_1}$, where $s = (s_1, s_2)$. \square

Now we can compute the homology of surgeries on S . In each case, we write down the d -invariants, which are the gradings of the top element in each $\mathbb{F}[[U]]$ summand.

Proposition 6.8. *Let Wh be the Whitehead link, $\Lambda = \text{diag}(p_1, p_2)$ and Y be the surgery manifold $S_\Lambda^3(Wh)$. Then $\text{Spin}^c(Y)$ can be identified with $\mathbb{Z}^2/\Lambda \cong \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$, so we use $(t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$ to denote the Spin^c structures over Y . Then, the Floer homology of Y is as follows.*

- If $p_1 = p_2 = 0$, then $\mathbf{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 4}, & (t_1, t_2) = (0, 0); \\ 0, & \text{otherwise,} \end{cases}$ with $d = -1, -1, 0, 0$.
- If $p_1 > 0, p_2 = 0$, then $\mathbf{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 2}, & (t_1, t_2) = (t_1, 0); \\ 0, & \text{otherwise.} \end{cases}$ Their d -invariants are $d(Y, (0, 0)) = \frac{p_1}{4} - \frac{7}{4}, \frac{p_1}{4} - \frac{3}{4}$, and $d(Y, (t_1, 0)) = \frac{(2s_1+p_1)^2}{4p_1} + \frac{1}{4}, \frac{(2s_1+p_1)^2}{4p_1} - \frac{3}{4}$, when $t_1 \neq 0$, where s_1 is an integer in the class $t_1 \in \mathbb{Z}/p_1\mathbb{Z}$ such that $-p_1 < s_1 \leq 0$.
- If $p_1 < 0, p_2 = 0$, then $\mathbf{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 2} \oplus (\mathbb{F}[[U]]/U), & (t_1, t_2) = (0, 0); \\ \mathbb{F}[[U]]^{\oplus 2}, & t_1 \neq 0, t_2 = 0; \\ 0, & \text{otherwise.} \end{cases}$ Their d -invariants are $d(Y, (t_1, 0)) = \frac{(2s_1-p_1)^2}{4p_1} + \frac{3}{4}, \frac{(2s_1-p_1)^2}{4p_1} - \frac{1}{4}$, where s_1 is an integer in the class $t_1 \in \mathbb{Z}/p_1\mathbb{Z}$ such that $p_1 < s_1 \leq 0$.
- If $p_1 > 0, p_2 > 0$, then $\mathbf{HF}^-(Y, (t_1, t_2)) = \mathbb{F}[[U]]$, $\forall (t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$. Their d -invariants are $d(Y, (0, 0)) = \frac{p_1+p_2-10}{4}$, and $d(Y, (t_1, t_2)) = \frac{(2s_1+p_1)^2}{4p_1} + \frac{(2s_2+p_2)^2}{4p_2} - \frac{1}{2}$, when $(t_1, t_2) \neq (0, 0)$, where s_i is an integer in the class $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ such that $-p_i < s_i \leq 0$.
- If $p_1 > 0, p_2 < 0$, then $\mathbf{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U), & (t_1, t_2) = (0, 0); \\ \mathbb{F}[[U]], & \text{otherwise.} \end{cases}$ Their d -invariants are $d(Y, (t_1, t_2)) = \frac{(2s_1+p_1)^2}{4p_1} + \frac{(2s_2-p_2)^2}{4p_2}$, where s_i is an integer in the class $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ such that $-|p_i| < s_i \leq 0$.
- If $p_1, p_2 < 0$, then $\mathbf{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U), & (t_1, t_2) = (0, 0); \\ \mathbb{F}[[U]], & \text{otherwise.} \end{cases}$ Their d -invariants are $d(Y, (t_1, t_2)) = \frac{(2s_1-p_1)^2}{4p_1} + \frac{(2s_2-p_2)^2}{4p_2} + \frac{1}{2}$, where s_i is an integer in the class $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ such that $p_i < s_i \leq 0$.

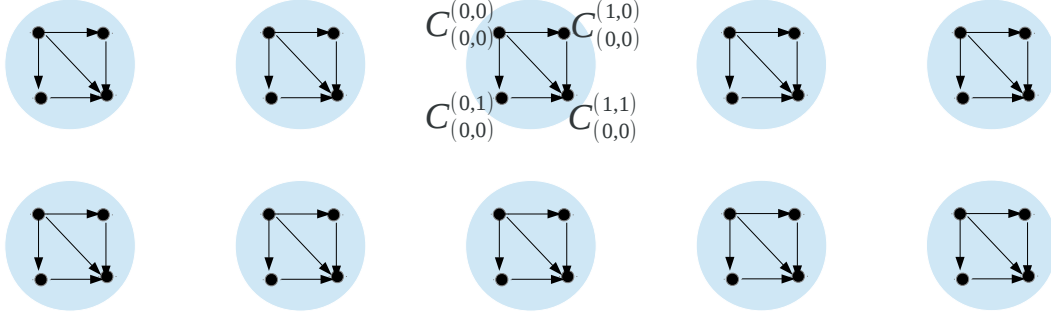


FIGURE 6.5. **The surgery complex for $\Lambda = (0,0)$.** Every dot represents a complex C_s^ε which is a certain generalized Floer complex $A_s^-(Wh)$, and every arrow represents a Φ -map according to the endpoints of the arrow. We only label the four complexes C_s^ε for the Spin^c structure $s = (0,0)$, and the others are similar.

Proof. First, let's look at the $(0,0)$ -surgery on the Whitehead link. The surgery complex splits into a direct product of squares of chain complexes according to Spin^c structures. See Figure 6.5. In the (s_1, s_2) Spin^c structure, the factor of the direct product is the following square of chain complexes:

$$\begin{array}{ccc}
 A_{s_1, s_2}^- & \xrightarrow{\Phi_{s_1, s_2}^{+L_1} + \Phi_{s_1, s_2}^{-L_1}} & A_{+\infty, s_2}^- \\
 \Phi_{s_1, s_2}^{+L_2} + \Phi_{s_1, s_2}^{-L_2} \downarrow & \searrow \sum \Phi_{s_1, s_2}^{\pm L_1 \cup \pm L_2} & \downarrow \Phi_{+\infty, s_2}^{+L_2} + \Phi_{+\infty, s_2}^{-L_2} \\
 A_{s_1, +\infty}^- & \xrightarrow{\Phi_{s_1, +\infty}^{+L_1} + \Phi_{s_1, +\infty}^{-L_1}} & A_{+\infty, +\infty}^-
 \end{array}$$

For the torsion Spin^c structure $(0,0) \in \mathbb{Z}^2$, since $\Phi_{0,0}^{+L_1} \simeq \Phi_{0,0}^{-L_1}$, $\Phi_{0,0}^{+L_2} \simeq \Phi_{0,0}^{-L_2}$, $\Phi_{0,+\infty}^{+L_1} \simeq \Phi_{0,\infty}^{-L_1}$, $\Phi_{+\infty,0}^{+L_2} \simeq \Phi_{+\infty,0}^{-L_2}$, the perturbed surgery complex is as follows:

$$\begin{array}{ccc}
 A_{0,0}^- & \xrightarrow{0} & A_{+\infty,0}^- \\
 0 \downarrow & & \downarrow 0 \\
 A_{0,+\infty}^- & \xrightarrow{0} & A_{+\infty,+\infty}^-
 \end{array}$$

Therefore, the homology is $\mathbb{F}[[U]]^{\oplus 4}$ generated by $d_1 \in C_{(0,0)}^{(0,0)}$, $d_3 \in C_{(0,0)}^{(1,0)}$, $d_1 \in C_{(0,0)}^{(0,1)}$, $d_1 \in C_{(0,0)}^{(1,1)}$. Since $c_1((0,0)) = (0,0)$, $\chi(W) = 2$, $\sigma(W) = 0$, from Equation (6.3), we also get their absolute gradings $\mu_{0,0}^{0,0}([d_1]) = -1$, $\mu_{0,0}^{1,0}([d_3]) = 0$, $\mu_{0,0}^{0,1}([d_1]) = 0$, $\mu_{0,0}^{1,1}([d_3]) = -1$.

For the non-torsion Spin^c structure $(s_1, s_2) \in \mathbb{Z}^2$, $s_1 > 0$, since $\Phi_{s_1, s_2}^{+L_1} + \Phi_{s_1, s_2}^{-L_1}$ acts on homology as $id + U^{s_1} \cdot id = (1 + U^{s_1}) \cdot id$, which is a quasi-isomorphism, it follows that the homology of this Spin^c structure is 0. Indeed, one can consider the horizontal filtration for this square, whose associated graded is the direct sum of the two acyclic horizontal rows. A similar argument applies to all the other non-torsion Spin^c structures.

Second, let's look at the $(p_1, 0)$ -surgery with $p_1 \neq 0$, which gives rise to a manifold with $b_1 = 1$. Suppose $p_1 > 0$. In order to compute the homology, we need some filtrations to kill acyclic sub-complexes and quotient complexes. Let $\mathcal{F}_1(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) = -s_1$, $\mathcal{F}_2(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) = s_1 - (\varepsilon_1 - 1)p_1$. Without loss of generality, see Figure 6.6 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (1,0)$.

For any $(t_1, t_2) \in \text{Spin}^c(Y) = \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}$ with $t_2 \neq 0$, the Floer homology is 0. Indeed, we can consider the union of all these Spin^c structures, which corresponds to the subcomplex

$$\mathcal{R}_1 = \bigoplus_{s_2 \neq 0} (C_{(s_1, s_2)}^{(0,0)} \oplus C_{(s_1, s_2)}^{(1,0)} \oplus C_{(s_1, s_2)}^{(0,1)} \oplus C_{(s_1, s_2)}^{(1,1)}).$$

Since $\Phi_{s_1, s_2}^{+L_2} + \Phi_{s_1, s_2}^{-L_2}, s_2 \neq 0$ acts on homology as $id + U^{|s_2|} \cdot id = (1 + U^{|s_2|}) \cdot id$, which is a quasi-isomorphism, the following square is acyclic:

$$\begin{array}{ccc} A_{s_1, s_2}^- & \xrightarrow{\Phi_{s_1, s_2}^{+L_1}} & A_{+\infty, s_2}^- \\ \Phi_{s_1, s_2}^{+L_2} + \Phi_{s_1, s_2}^{-L_2} \downarrow & \searrow \Phi_{s_1, s_2}^{+L_1 \cup +L_2} + \Phi_{s_1, s_2}^{+L_1 \cup -L_2} & \downarrow \Phi_{+\infty, s_2}^{+L_2} + \Phi_{+\infty, s_2}^{-L_2} \\ A_{s_1, +\infty}^- & \xrightarrow{\Phi_{s_1, +\infty}^{+L_1}} & A_{+\infty, +\infty}^- \end{array}$$

The associated graded complex of \mathcal{F}_1 splits as a direct product of the above squares, so \mathcal{R}_1 is acyclic.

For the Spin^c structure $(t_1, 0)$, we first kill the acyclic subcomplex

$$\mathcal{R}_2 = \bigoplus_{s_1 > 0} C_{(s_1, 0)}^{(\varepsilon_1, \varepsilon_2)}.$$

Since the inclusion map $I_{s_1, 0}^{+L_1}$ is id for all $s_1 > 0$, the associated graded complex of the filtration \mathcal{F}_1 splits as a direct product of acyclic complexes in the form of

$$\begin{array}{ccc} C_{(s_1, 0)}^{(0,0)} & \xrightarrow{\Phi_{s_1, 0}^{+L_1}} & C_{(s_1, 0)}^{(1,0)} \\ \Phi_{s_1, 0}^{+L_2} + \Phi_{s_1, 0}^{-L_2} \downarrow & \searrow \Phi_{s_1, s_2}^{+L_1 \cup +L_2} + \Phi_{s_1, s_2}^{+L_1 \cup -L_2} & \downarrow \Phi_{+\infty, 0}^{+L_2} + \Phi_{+\infty, 0}^{-L_2} \\ C_{(s_1, 0)}^{(0,1)} & \xrightarrow{\Phi_{s_1, +\infty}^{+L_1}} & C_{(s_1, 0)}^{(1,1)}. \end{array}$$

Thus \mathcal{R}_2 is acyclic.

On the other hand, we have another acyclic subcomplex

$$\mathcal{R}_3 = \bigoplus_{\mathcal{F}_2 \leq 0} C_{(s_1, 0)}^{(\varepsilon_1, \varepsilon_2)}.$$

In fact, since the inclusion maps $I_{s_1, 0}^{-L_1}$ and $I_{s_1, +\infty}^{-L_1}$ are both id when $s_1 < 0$, the associated graded complex of the filtration \mathcal{F}_2 splits as a direct product of acyclic complexes in the form of

$$\begin{array}{ccc} C_{(s_1, 0)}^{(0,0)} & \xrightarrow{\Phi_{s_1, 0}^{-L_1}} & C_{(s_1 + p_1, 0)}^{(1,0)} \\ \Phi_{s_1, 0}^{+L_2} + \Phi_{s_1, 0}^{-L_2} \downarrow & \searrow \Phi_{s_1, s_2}^{-L_1 \cup +L_2} + \Phi_{s_1, s_2}^{-L_1 \cup -L_2} & \downarrow \Phi_{+\infty, 0}^{+L_2} + \Phi_{+\infty, 0}^{-L_2} \\ C_{(s_1, 0)}^{(0,1)} & \xrightarrow{\Phi_{s_1, +\infty}^{-L_1}} & C_{(s_1 + p_1, 0)}^{(1,1)}. \end{array}$$

Thus \mathcal{R}_3 is acyclic. So the quotient complex $\mathcal{Q} = \mathcal{C}^- / \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ is a direct product of

$$C_{(s_1, 0)}^{(0,0)} \xrightarrow{\Phi_{s_1, 0}^{+L_2} + \Phi_{s_1, 0}^{-L_2}} C_{(s_1, 0)}^{(0,1)},$$

where $-p_1 + 1 \leq s_1 \leq 0$. From the computations of the inclusion maps, we know that $\Phi_{s_1, 0}^{+L_2} \simeq \Phi_{s_1, 0}^{-L_2}$. Thus the homology of each Spin^c structure $(t_1, 0) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}$ is $\mathbb{F}[[U]]^{\oplus 2}$. Note that $\chi(W) = 2, \sigma(W) = 1$. When $-p_1 + 1 \leq s_1 < 0$, the complex $C_{(s_1, 0)}^{(0,0)} = A_{s_1, 0}^-$ has a_2 as a generator of its

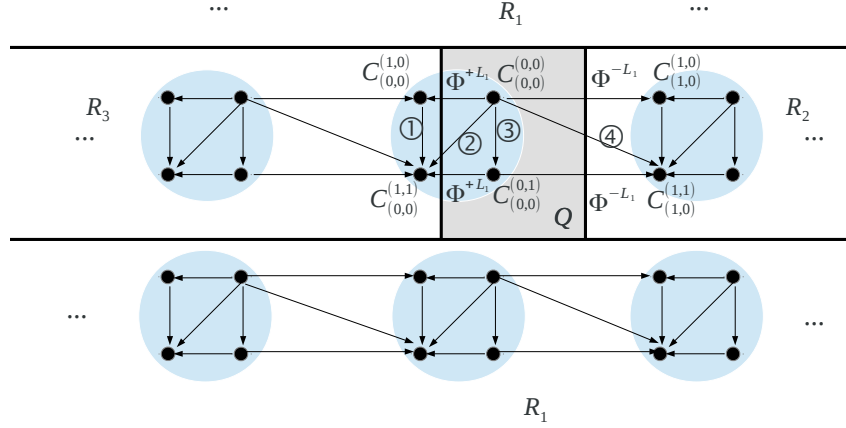


FIGURE 6.6. **The surgery complex for $\Lambda = (1, 0)$.** Every dot represents a complex C_s^ϵ which is a certain generalized Floer complex $A_s^-(Wh)$, and in every shaded circle the complexes C_s^ϵ 's have the same subscript s . Every arrow represents a Φ -map according to the endpoints of the arrow, where we omit the subscripts. All the parallel arrows share the same type of $\Phi^{\vec{M}}$, i.e. having the same superscript \vec{M} . The arrows with circled numbers 1, 2, 3, 4 are $\Phi_{+\infty,0}^{+L_2} + \Phi_{+\infty,0}^{-L_2}$, $\Phi_{0,0}^{+L_2 \cup +L_1} + \Phi_{0,0}^{-L_2 \cup +L_1}$, $\Phi_{0,0}^{+L_2} + \Phi_{0,0}^{-L_2}$, and $\Phi_{0,0}^{+L_2 \cup -L_1} + \Phi_{0,0}^{-L_2 \cup -L_1}$ respectively. The regions R_1, R_2, R_3 divided by the (thicker) line are corresponded to the acyclic subcomplexes $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$. The shaded region Q corresponds to the truncated complex \mathcal{Q} .

homology of grading $\mu_{s_1,0}^{0,0}(a_2) = \frac{s_1^2}{p_1} + s_1 + \frac{p_1}{4} + \frac{1}{4}$, and the complex $C_{(s_1,0)}^{(0,1)} = A_{s_1,+\infty}^-$ has d_1 as a generator of its homology of grading $\mu_{s_1,0}^{0,1}(d_1) = \frac{s_1^2}{p_1} + s_1 + \frac{p_1}{4} - \frac{3}{4}$. While for the $(0,0)$ Spin^c structure, $C_{(0,0)}^{(0,0)} = A_{0,0}^-$ has d_1 as a generator of its homology with grading $\mu_{0,0}^{0,0}(d_1) = \frac{p_1}{4} - \frac{7}{4}$, and $C_{(0,0)}^{(0,1)} = A_{0,+\infty}^-$ has d_1 as a generator of its homology with grading $\mu_{0,0}^{0,1}(d_1) = \frac{p_1}{4} - \frac{3}{4}$.

The case of $p_1 < 0$ is similar. We first kill the acyclic subcomplex

$$\mathcal{R}_1 = \bigoplus_{s_2 \neq 0} (C_{(s_1,s_2)}^{(0,0)} \oplus C_{(s_1,s_2)}^{(1,0)} \oplus C_{(s_1,s_2)}^{(0,1)} \oplus C_{(s_1,s_2)}^{(1,1)}).$$

Thus, the homology for the Spin^c structure (t_1, t_2) with $t_2 \neq 0$ is 0. Next, we kill the acyclic quotient complexes

$$\mathcal{R}_2 = \bigoplus_{s_1 > 0} C_{(s_1,0)}^{(\varepsilon_1, \varepsilon_2)}, \quad \mathcal{R}_3 = \bigoplus_{s_1 - \varepsilon_1 p_1 < 0} C_{(s_1,0)}^{(\varepsilon_1, \varepsilon_2)}.$$

In the $(0,0)$ Spin^c structure, the remaining complexes are as follows

$$\begin{array}{ccccc} C_{(p_1,0)}^{(1,0)} & \xleftarrow{\Phi_{0,0}^{-L_1}} & C_{(0,0)}^{(0,0)} & \xrightarrow{\Phi_{0,0}^{+L_1}} & C_{(0,0)}^{(1,0)} \\ \Phi_{+\infty,0}^{+L_2} + \Phi_{+\infty,0}^{-L_2} \downarrow & \Phi_{0,0}^{-L_1 \cup +L_2} + \Phi_{0,0}^{-L_1 \cup -L_2} \swarrow & \Phi_{0,0}^{+L_2} + \Phi_{0,0}^{-L_2} \downarrow & \Phi_{0,0}^{+L_1 \cup +L_2} + \Phi_{0,0}^{+L_1 \cup -L_2} \searrow & \Phi_{+\infty,0}^{+L_2} + \Phi_{+\infty,0}^{-L_2} \downarrow \\ C_{(p_1,0)}^{(1,1)} & \xleftarrow{\Phi_{0,0}^{-L_1}} & C_{(0,0)}^{(0,1)} & \xrightarrow{\Phi_{0,0}^{+L_1}} & C_{(0,0)}^{(1,1)} \end{array}$$

Since $\Phi_{+\infty,0}^{+L_2}, \Phi_{+\infty,0}^{-L_2}$ are chain homotopic, we can replace $\Phi_{+\infty,0}^{+L_2} + \Phi_{+\infty,0}^{-L_2}$ by 0 in the perturbed surgery complex. Therefore, we can also replace the diagonal maps by 0. Thus, we have two split

complexes,

$$C_{(p_1,0)}^{(1,0)} \xleftarrow{\Phi_{0,0}^{-L_1}} C_{(0,0)}^{(0,0)} \xrightarrow{\Phi_{0,0}^{+L_1}} C_{(0,0)}^{(1,0)},$$

$$C_{(p_1,0)}^{(1,1)} \xleftarrow{\Phi_{0,+\infty}^{-L_1}} C_{(0,0)}^{(0,1)} \xrightarrow{\Phi_{0,+\infty}^{+L_1}} C_{(0,0)}^{(1,1)}.$$

Since $C_{(p_1,0)}^{(1,0)} = C_{(0,0)}^{(1,0)}$ and $\Phi_{0,0}^{+L_1} \simeq \Phi_{0,0}^{-L_1}$, we can replace $\Phi_{0,0}^{+L_1}$ by $\Phi_{0,0}^{-L_1}$ in the perturbed complex. By changing basis, we can split the first row as $\text{cone}(\Phi_{0,0}^{+L_1}) \oplus C_{(p_1,0)}^{(1,0)}$ with homology $\mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U)$. Similarly, from that $\Phi_{0,+\infty}^{\pm L_1}$ are quasi-isomorphisms, it follows that the second row is quasi-isomorphic to $C_{(0,0)}^{(1,1)}$ by changing basis. Thus in the $(0,0)$ Spin^c structure, the homology is $\mathbb{F}[[U]]^{\oplus 2} \oplus (\mathbb{F}[[U]]/U)$.

For the other Spin^c structures $(t_1, 0), t_1 \neq 0$, the remaining complexes are as follows

$$C_{(s_1,0)}^{(1,0)} \xrightarrow{\Phi_{+\infty,0}^{+L_2} + \Phi_{+\infty,0}^{-L_2}} C_{(s_1,0)}^{(1,1)},$$

where the integer s_1 is in the residue class $t_1 \in \mathbb{Z}/p_1\mathbb{Z}$ such that $p_1 < s_1 < 0$. Similarly, we replace $\Phi_{+\infty,0}^{+L_2} + \Phi_{+\infty,0}^{-L_2}$ by 0, and get that the homology is $\mathbb{F}[[U]]^{\oplus 2}$. The correction terms can be computed similarly with $\chi(W) = 2, \sigma(W) = -1$.

Finally, let's look at the (p_1, p_2) -surgery, where $p_1 p_2 \neq 0$. This breaks down to three cases: $p_1, p_2 > 0$, $p_1, p_2 < 0$, and $p_1 p_2 < 0$. We apply the truncation tricks shown in [5] Section 8.3.

(1) When $p_1, p_2 > 0$, the (p_1, p_2) -surgery is actually a large surgery, so its homology can be derived from A_s^- directly. However, we still compute them by elementary methods. We construct two filtrations,

$$\mathcal{F}_{00}(C_{(s_1,s_2)}^{(\varepsilon_1,\varepsilon_2)}) = -s_1 - s_2, \quad \mathcal{F}_{11}(C_{(s_1,s_2)}^{(\varepsilon_1,\varepsilon_2)}) = s_1 - (\varepsilon_1 - 1)p_1 + s_2 - (\varepsilon_2 - 1)p_2.$$

Without loss of generality, see Figure 6.7 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (1, 1)$.

We first consider an acyclic subcomplex

$$\mathcal{R}_1 = \bigoplus_{\max\{s_1, s_2\} > 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}.$$

In fact, since the inclusion maps $I_{s_1, s_2}^{+L_1}, s_1 > 0$ and $I_{s_1, s_2}^{+L_2}, s_2 > 0$ are both id 's, the associated graded complex of the filtration \mathcal{F}_{00} splits as a direct product of acyclic squares $R_{s,0,0}$ in Equation 4.2:

$$\begin{array}{ccc} A_{s_1, s_2}^- & \xrightarrow{\Phi_{s_1, s_2}^{+L_1}} & A_{+\infty, s_2}^- \\ \Phi_{s_1, s_2}^{+L_2} \downarrow & \searrow \Phi_{s_1, s_2}^{+L_1 \cup +L_2} & \downarrow \Phi_{+\infty, s_2}^{+L_2} \\ A_{s_1, +\infty}^- & \xrightarrow{\Phi_{s_1, +\infty}^{+L_1}} & A_{+\infty, +\infty}^- \end{array}$$

Thus \mathcal{R}_1 is acyclic.

There is another acyclic subcomplex

$$\mathcal{R}_2 = \bigoplus_{\max\{s_1 - (\varepsilon_1 - 1)p_1, s_2 - (\varepsilon_2 - 1)p_2\} \leq 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}.$$

One can directly check \mathcal{R}_2 is a subcomplex by computation. Because the inclusion maps $I_{s_1, s_2}^{-L_1}, s_1 < 0$ and $I_{s_1, s_2}^{-L_2}, s_2 < 0$ are both *id*'s, the associated graded complex of \mathcal{F}_{11} splits as a product of acyclic squares $R_{s,1,1}$:

$$\begin{array}{ccc} A_{s_1, s_2}^- & \xrightarrow{\Phi_{s_1, s_2}^{-L_1}} & A_{+\infty, s_2}^- \\ \Phi_{s_1, s_2}^{-L_2} \downarrow & \searrow \Phi_{s_1, s_2}^{-L_1 \cup -L_2} & \downarrow \Phi_{+\infty, s_2}^{-L_2} \\ A_{s_1, +\infty}^- & \xrightarrow{\Phi_{s_1, +\infty}^{-L_1}} & A_{+\infty, +\infty}^- \end{array}$$

where $s_1 + p_1 \leq 0, s_2 + p_2 \leq 0$. Thus \mathcal{R}_2 is acyclic.

Let $\mathcal{C}_1 = \mathcal{C}/(\mathcal{R}_1 + \mathcal{R}_2)$. Inside \mathcal{C}_1 , there are two acyclic subcomplexes

$$\begin{aligned} \mathcal{R}_3 &= \left\{ \bigoplus_{s_1 - (\varepsilon_1 - 1)p_1 \leq 0, -p_2 + 1 \leq s_2 \leq 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)} \right\} \cap \mathcal{C}_1 = \bigoplus_{s_1 + p_1 \leq 0, 0 \geq s_2 > -p_2} (C_{(s_1, s_2)}^{(0,0)} \oplus C_{(s_1 + p_1, s_2)}^{(1,0)}), \\ \mathcal{R}_4 &= \left\{ \bigoplus_{s_2 - (\varepsilon_2 - 1)p_2 \leq 0, -p_1 + 1 \leq s_1 \leq 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)} \right\} \cap \mathcal{C}_1 = \bigoplus_{s_2 + p_2 \leq 0, 0 \geq s_1 > -p_1} (C_{(s_1, s_2)}^{(0,0)} \oplus C_{(s_1, s_2 + p_2)}^{(1,1)}). \end{aligned}$$

In fact, the associated graded complex of \mathcal{F}_{11} on \mathcal{R}_3 splits as a direct product of acyclic complexes $C_{(s_1, s_2)}^{(0,0)} \xrightarrow{\Phi_{s_1, s_2}^{-L_1}} C_{(s_1 + p_1, s_2)}^{(1,0)}$, because the inclusion map $I_{s_1, s_2}^{-L_1}, s_1 < 0$ is *id*. Thus \mathcal{R}_3 is acyclic. Similar argument applies to \mathcal{R}_4 .

At last, we look at the quotient complex

$$\mathcal{Q} = \mathcal{C}_1/(\mathcal{R}_3 + \mathcal{R}_4) = \bigoplus_{-p_1 < s_1 \leq 0, -p_2 < s_2 \leq 0} C_{(s_1, s_2)}^{(0,0)},$$

where $C_{(s_1, s_2)}^{(0,0)} = A_{s_1, s_2}^-$. There is only one A_s^- left in each Spin^c structure Y with homology $\mathbb{F}[[U]]$. For $(s_1, s_2) = (0, 0)$, the complex $C_{(0,0)}^{(0,0)} = A_{0,0}^-$ has d_1 as a generator of its homology with grading $\mu_{0,0}^{0,0}(d_1) = \frac{p_1 + p_2 - 10}{4}$. For $-p_1 < s_1 < 0$, the complex $C_{(s_1, s_2)}^{(0,0)} = A_{s_1, s_2}^-$ has a_2 as a generator of its homology with grading $\mu_{s_1, s_2}^{0,0}(a_2) = \frac{s_1^2}{p_1} + \frac{s_2^2}{p_2} + s_1 + s_2 + \frac{p_1 + p_2 - 2}{4}$. Similarly, we have the same formula for $-p_2 < s_2 < 0, -p_1 < s_1 \leq 0$.

(2) When $p_1 p_2 < 0$, we might as well suppose $p_1 > 0, p_2 < 0$ due to the symmetry of the two components. We construct four filtrations

$$\begin{aligned} \mathcal{F}_{00}(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) &= -s_1 + s_2, \quad \mathcal{F}_{01}(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) = -s_1 - s_2 + (\varepsilon_2 - 1)p_2, \\ \mathcal{F}_{10}(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) &= s_1 - (\varepsilon_1 - 1)p_1 + s_2, \quad \mathcal{F}_{11}(C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}) = s_1 - (\varepsilon_1 - 1)p_1 - s_2 + (\varepsilon_2 - 1)p_2. \end{aligned}$$

Without loss of generality, see Figure 6.8 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (1, -1)$. We first kill an acyclic subcomplex \mathcal{R}_1 composed of $C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}$ with $s_1 > 0$. Indeed, the associated graded complex of \mathcal{F}_{00} on \mathcal{R}_1 splits as a direct product of acyclic squares, since the inclusion map $I_{s_1, s_2}^{+L_1}, s_1 > 0$ is *id*.

We have another acyclic subcomplex

$$\mathcal{R}_2 = \bigoplus_{s_1 - (\varepsilon_1 - 1)p_1 \leq 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}.$$

In fact, since $\Phi_{s_1, s_2}^{-L_1}$ are quasi-isomorphisms when $s_1 < 0$, the associated graded of the filtration \mathcal{F}_{10} for \mathcal{R}_2 splits as a direct product of acyclic squares $R_{s,1,0}$. Thus \mathcal{R}_2 is acyclic.

Thus, \mathcal{C} is quasi-isomorphic to the quotient complex

$$\mathcal{C}_1 = \mathcal{C}/(\mathcal{R}_1 + \mathcal{R}_2) = \bigoplus_{-p_1 < s_1 \leq 0} (C_{(s_1, s_2)}^{(0,0)} \oplus C_{(s_1, s_2)}^{(0,1)}).$$

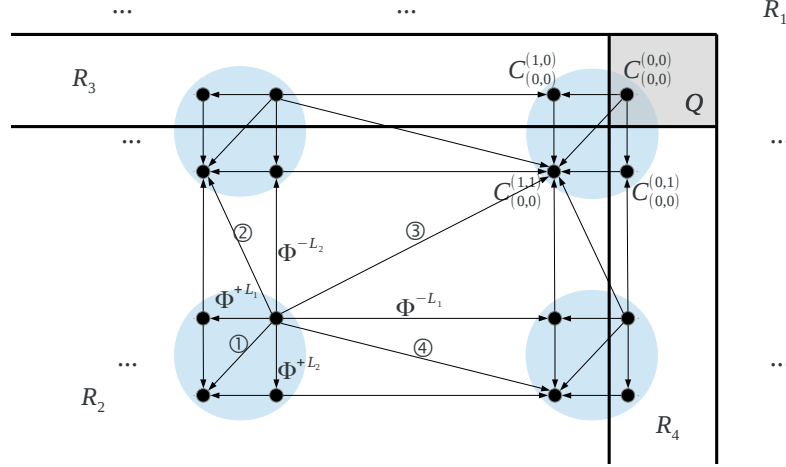


FIGURE 6.7. **The surgery complex for $\Lambda = (1, 1)$.** The arrows with circled numbers 1, 2, 3, 4 are $\Phi^{+L_1 \cup L_2}$, $\Phi^{+L_1 \cup -L_2}$, $\Phi^{-L_1 \cup -L_2}$, and $\Phi^{-L_1 \cup L_2}$ respectively. The regions R_1, R_2, R_3, R_4 divided by the (thicker) line are corresponded to the acyclic subcomplexes $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$. The shaded region Q corresponds to the truncated complex \mathcal{Q} .

We have an acyclic quotient complex \mathcal{R}_3 of \mathcal{C}_1

$$\mathcal{R}_3 = \bigoplus_{-p_1 < s_1 \leq 0, s_2 > 0} (C_{(s_1, s_2)}^{(0,0)} \oplus C_{(s_1, s_2)}^{(0,1)}),$$

since the inclusion maps $I_{s_1, s_2}^{+L_2}, s_2 > 0$ are all the identities. Furthermore, we have another acyclic quotient complex \mathcal{R}_4 of \mathcal{C}_1

$$\mathcal{R}_4 = \bigoplus_{-p_1 < s_1 \leq 0, s_2 < 0} (C_{(s_1, s_2)}^{(0,0)} \oplus C_{(s_1, s_2 - p_2)}^{(0,1)}).$$

Thus \mathcal{C} is quasi-isomorphic to

$$\mathcal{Q} = \mathcal{C}_1 \setminus (\mathcal{R}_3 \cup \mathcal{R}_4) = \left\{ \bigoplus_{-p_1 < s_1 \leq 0} (C_{(s_1, 0)}^{(0,0)} \oplus C_{(s_1, 0)}^{(0,1)} \oplus C_{(s_1, p_2)}^{(0,1)}) \right\} \oplus \left\{ \bigoplus_{-p_1 < s_1 \leq 0, p_2 < s_2 < 0} C_{(s_1, s_2)}^{(0,1)} \right\}.$$

In the Spin^c structure $(t_1, 0) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$, we have the complex as follows,

$$\begin{aligned} C_{(s_1, 0)}^{(0,0)} = A_{s_1, 0}^- &\xrightarrow{\Phi_{s_1, 0}^{+L_2}} A_{s_1, +\infty}^- = C_{(s_1, 0)}^{(0,1)} \\ &\searrow \Phi_{s_1, 0}^{-L_2} \\ &A_{s_1, +\infty}^- = C_{(s_1, p_2)}^{(0,1)}, \end{aligned}$$

where s_1 is an integer such that $-p_1 < s_1 \leq 0$ and $s_1 \equiv t_1 \pmod{p_1}$. Since the inclusion maps $I_{0,0}^{\pm L_2}$ induce the same action on homology, $\Phi_{0,0}^{\pm L_2}$ are chain homotopic to each other. By Corollary 5.3, we can replace $A_{0,0}^-, A_{0,+ \infty}^-$ by the complex $\mathbb{F}[[U_1, U_2]] \xrightarrow{U_1 - U_2} \mathbb{F}[[U_1, U_2]]$, where the generators are g_1, g_2 . Then, we can replace the chain maps $I_{0,0}^{\pm L_2}$ by the same chain map \tilde{I} , where $\tilde{I}(g_i) = U_1 g_i$. Thus, the homology of the $(0, 0)$ Spin^c structure can be computed by this perturbed complex, which is $\mathbb{F}[[U]] \oplus \mathbb{F}[[U]]/U$. From above computation, the generator corresponding to $\mathbb{F}[[U]]$ is actually the generator of $H_*(C_{(0,0)}^{(0,1)})$, which is $d_1 \in A_{s_1, +\infty}^-$ with grading $\mu_{0,0}^{0,1}(d_1) = \frac{p_1 + p_2}{4}$ by Equation (6.3).

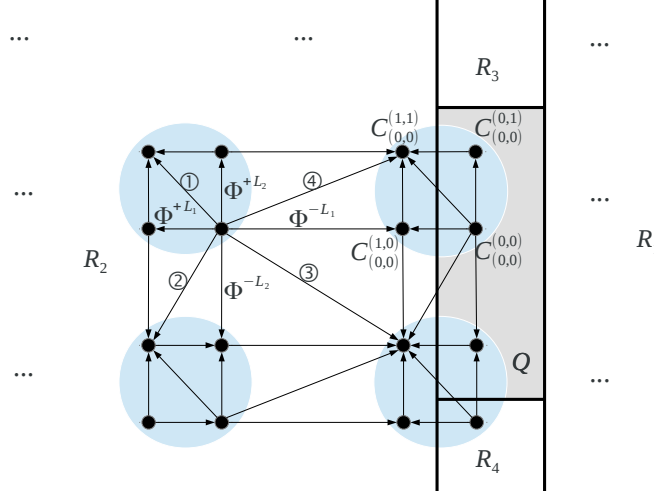


FIGURE 6.8. **The surgery complex for $\Lambda = (1, -1)$.** The arrows with circled numbers 1, 2, 3, 4 are $\Phi^{+L_1 \cup +L_2}$, $\Phi^{+L_1 \cup -L_2}$, $\Phi^{-L_1 \cup -L_2}$, and $\Phi^{-L_1 \cup +L_2}$ respectively. The regions R_1, R_2, R_3, R_4 divided by the (thicker) line are corresponded to the acyclic subcomplexes $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$. The shaded region Q corresponds to the truncated complex \mathcal{Q} .

On the other hand, since the inclusion maps $I_{s_1,0}^{-L_2}, s_1 < 0$ are all quasi-isomorphisms, we can kill the acyclic quotient complex $A_{s_1,0}^- \xrightarrow{\Phi^{-L_2}} A_{s_1,+\infty}^-$. Thus, the homology for the Spin^c structure $(t_1, 0) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$ with $t_1 \neq 0$ is $\mathbb{F}[[U]]$ generated by $d_1 \in A_{s_1,+\infty}^-$ of grading $\mu_{s_1,0}^{0,1}(d_1) = \frac{s_1^2}{p_1} + s_1 + \frac{p_1+p_2}{4}$, where s_1 is an integer with $-p_1 < s_1 < 0$ in the class t_1 modulo p_1 .

In every other Spin^c structure in the complex \mathcal{Q} , there is only one complex $C_{(s_1,s_2)}^{(0,1)} = A_{s_1,+\infty}^-$, $-p_1 < s_1 \leq 0$ of homology $\mathbb{F}[[U]]$ of grading $\frac{s_1^2}{p_1} + \frac{s_2^2}{p_2} + s_1 - s_2 + \frac{p_1+p_2}{4}$.

(3) The last case is when p_1, p_2 are both negative integers. We use two filtrations

$$\mathcal{F}_{00}(C_{(s_1,s_2)}^{(\varepsilon_1,\varepsilon_2)}) = s_1 + s_2, \quad \mathcal{F}_{11}(C_{(s_1,s_2)}^{(\varepsilon_1,\varepsilon_2)}) = -s_1 + (\varepsilon_1 - 1)p_1 - s_2 + (\varepsilon_2 - 1)p_2.$$

Without loss of generality, see Figure 6.9 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (-1, -1)$.

We first kill an acyclic quotient complex

$$\mathcal{R}_1 = \bigoplus_{\max\{s_1,s_2\} > 0} C_{(s_1,s_2)}^{(\varepsilon_1,\varepsilon_2)}.$$

By considering the filtration \mathcal{F}_{00} , we can see that \mathcal{R}_1 is acyclic. We also have another acyclic quotient complex

$$\mathcal{R}_2 = \bigoplus_{\min\{s_1 - \varepsilon_1 p_1, s_2 - \varepsilon_2 p_2\} < 0} C_{(s_1,s_2)}^{(\varepsilon_1,\varepsilon_2)}.$$

In fact, the inclusion maps $I_{s_1,s_2}^{-L_i}, s_i < 0$ are quasi-isomorphisms. Thus the associated graded complex of the filtration \mathcal{F}_{11} splits as a direct product of acyclic complexes

$$R_{s,1,1} \cap (\mathcal{C} \setminus \mathcal{R}_1)$$

where $\min\{s_1, s_2\} < 0$ and $R_{s,1,1}$ is in Equation (4.5). Therefore \mathcal{R}_2 is acyclic.

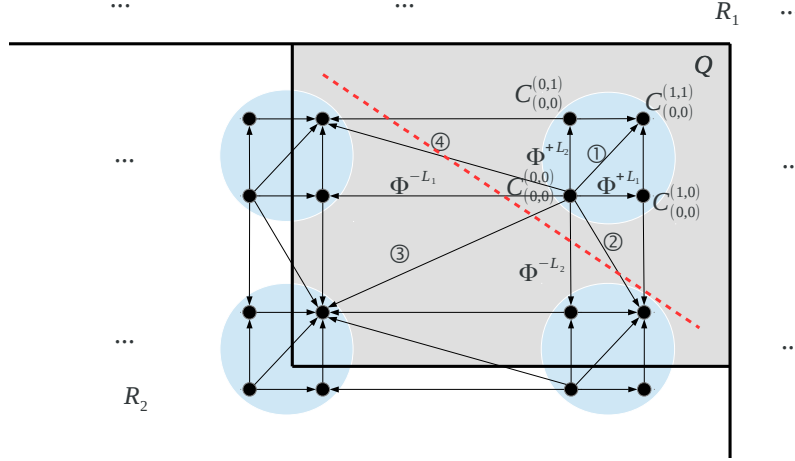


FIGURE 6.9. **The surgery complex for $\Lambda = (-1, -1)$.** The arrows with circled numbers 1, 2, 3, 4 are $\Phi^{+L_1 \cup +L_2}$, $\Phi^{+L_1 \cup -L_2}$, $\Phi^{-L_1 \cup -L_2}$, and $\Phi^{-L_1 \cup +L_2}$ respectively. The regions R_1, R_2 divided by the (thicker) line are corresponded to the acyclic subcomplexes $\mathcal{R}_1, \mathcal{R}_2$. The shaded region Q corresponds to the truncated complex \mathcal{Q} . The dashed (red) line indicates the filtration which gives rise to the mapping cone used in the argument, where A lies above the dashed line and B lies below the dashed line.

Hence, the subcomplex $\mathcal{Q} = \mathcal{C} \setminus (\mathcal{R}_1 \cup \mathcal{R}_2)$ is quasi-isomorphic to \mathcal{C} , where

$$\mathcal{Q} = \bigoplus_{\max\{s_1, s_2\} \leq 0, \min\{s_1 - \varepsilon_1 p_1, s_2 - \varepsilon_2 p_2\} \geq 0} C_{(s_1, s_2)}^{(\varepsilon_1, \varepsilon_2)}.$$

In the Spin^c structure (t_1, t_2) , $t_1 \neq 0, t_2 \neq 0$, there is only one complex $C_{(s_1, s_2)}^{(1, 1)}$ in \mathcal{Q} , thus having the homology $\mathbb{F}[[U]]$ with grading $\frac{(2s_1 - p_1)^2}{4p_1} + \frac{(2s_2 - p_2)^2}{4p_2} + \frac{1}{2}$, where s_1, s_2 are negative integers in the residue classes t_1, t_2 such that $s_i \geq p_i + 1, i = 1, 2$. In the Spin^c structure $(0, t_2)$, $t_2 \neq 0$, there are three complexes $C_{(0, s_2)}^{(0, 1)}, C_{(0, s_2)}^{(1, 1)}, C_{(p_1, s_2)}^{(1, 1)}$ in \mathcal{Q} , where s_2 is an integer in the residue class t_2 such that $p_2 < s_2 < 0$. Since the inclusion map $I_{0, s_2}^{\pm L_1}, s_2 \neq 0$ are quasi-isomorphisms, we can replace $\Phi_{0, s_2}^{+L_1}$ by $\Phi_{0, s_2}^{-L_1}$ in the perturbed complex and thus split it as a direct sum, $\text{cone}(\Phi_{0, s_2}^{+L_1}) \oplus C_{(0, s_2)}^{(1, 1)}$, by changing the basis. Thus the homology is the same as the homology of $C_{(0, s_2)}^{(1, 1)} = A_{+\infty, +\infty}^-$, which is $\mathbb{F}[[U]]$ generated by $[d_1]$ with grading $\frac{p_1}{4} + \frac{(2s_2 - p_2)^2}{4p_2} + \frac{1}{2}$. It is similar for $(t_1, 0) \in \text{Spin}^c(Y), t_1 \neq 0$.

The most interesting Spin^c structure is $(0, 0)$. It consists of nine complexes, which are also illustrated in Figure 6.9. By Proposition 5.2 and Corollary 5.3, in the perturbed complex we can replace all the A_s^- by the mapping cones $\mathbb{F}[[U_1, U_2]] \xrightarrow{U_1 - U_2} \mathbb{F}[[U_1, U_2]]$ and replace the edge maps by the corresponding maps. To compute the homology, we view it as a mapping cone: $\text{cone}(A \rightarrow B)$, where A, B are the quotient complex and subcomplex induced by a filtration illustrated in Figure 6.9. It is not hard to compute $H_*(A) = \mathbb{F}[[U]]/U$ and $H_*(B) = \mathbb{F}[[U]]$. Since $\mathbb{F}[[U]]/U$ is a torsion module, the homology long exact sequence descends to a short exact sequence

$$0 \rightarrow \mathbb{F}[[U]] \xrightarrow{i_*} \mathbf{HF}^-(S_\Lambda^3(L), (0, 0)) \rightarrow \mathbb{F}[[U]]/U \rightarrow 0.$$

Because $\text{Ext}_{\mathbb{F}[[U]]}^1(\mathbb{F}[[U]]/U, \mathbb{F}[[U]]) = \mathbb{F}[[U]]/U$ has two elements, the module $\mathbf{HF}^-(S_\Lambda^3(Wh), (0, 0))$ can either be $\mathbb{F}[[U]]$ or $(\mathbb{F}[[U]]/U) \oplus \mathbb{F}[[U]]$. By directly computing

$$H_*(\hat{\mathcal{C}}(\mathcal{H}, \Lambda, (0, 0)), \hat{\mathcal{D}}) = \widehat{\mathbf{HF}}(S_\Lambda^3(Wh), (0, 0)) \otimes \mathbb{F}^{\oplus 2},$$

we get that $H_*(\hat{\mathcal{C}}(\mathcal{H}, \Lambda, (0, 0)), \hat{\mathcal{D}}) = \mathbb{F}^{\oplus 6}$, thus $\widehat{\mathbf{HF}}(S_\Lambda^3(Wh), (0, 0)) = \mathbb{F}^{\oplus 3}$. Consequently, it follows that

$$\mathbf{HF}^-(S_\Lambda^3(Wh), (0, 0)) = \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U),$$

when $\Lambda = \text{diag}(p_1, p_2)$ with $p_1, p_2 < 0$. Thus, $i_*(1) = (1, 0)$ or $(1, 1)$. Thereby, $[d_1] = 1 \in H_*(C_{(p_1, p_2)}^{(1, 1)})$ is also a generator of $\mathbf{HF}^-(S_\Lambda^3(L), (0, 0))$ of grading $\frac{p_1 + p_2 + 2}{4}$. \square

Theorem 6.9. *Let \vec{L} be the two-bridge link $b(8k, 4k + 1)$, $k \in \mathbb{N}$ and $\Lambda = \text{diag}(p_1, p_2)$, $p_1, p_2 \in \mathbb{Z}$ be the framing matrix of an integer surgery on \vec{L} . As in Proposition 6.8, we use $(t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$ to denote the Spin^c structures over $S_\Lambda^3(\vec{L})$. Then, we have the Floer homology*

$$(6.4) \quad \mathbf{HF}^-(S_\Lambda^3(\vec{L}), (t_1, t_2)) = \begin{cases} \mathbf{HF}^-(S_\Lambda^3(Wh), (0, 0)) \oplus \mathbb{F}^{\oplus(k-1)}, & (t_1, t_2) = (0, 0), \\ \mathbf{HF}^-(S_\Lambda^3(Wh), (t_1, t_2)), & \text{otherwise.} \end{cases}$$

The correction terms of the elements in the $\mathbf{HF}^-(S_\Lambda^3(Wh))$ -summand are the same as in $\mathbf{HF}^-(S_\Lambda^3(Wh))$.

Proof. By Proposition 6.4, $CFL^-(\vec{L}) = CFL^-(Wh) \oplus \bigoplus_{i=1}^{k-1} (N, \partial^-)$. Let $\mathcal{N} = \bigoplus_{i=1}^{k-1} (N, \partial^-)$. We define \mathcal{N}_s similarly as A_s^- in (2.2). Concretely, suppose G be a set of homogeneous generators of \mathcal{N} as a $\mathbb{F}[[U_1, U_2]]$ -module, and for $x \in G$,

$$\partial x = \sum_{y \in G} k_{xy} y,$$

where $k_{xy} \in \mathbb{F}[[U_1, U_2]]$. Let $A(x) = (A_1(x), A_2(x))$ denote the Alexander filtration of $x \in G$. Define \mathcal{N}_s by

$$\partial x = \sum_{y \in G} k_{xy} \cdot U_1^{\max\{A_1(x) - s_1, 0\} - \max\{A_1(y) - s_1, 0\}} U_2^{\max\{A_2(x) - s_2, 0\} - \max\{A_2(y) - s_2, 0\}} \cdot y.$$

Thus $A_s^-(\vec{L}) = A_s^-(Wh) \oplus \mathcal{N}_s$. Thus all the inclusion maps $I^{\pm L_i}, i = 1, 2$ preserve this direct sum decomposition. Since the complexes $\mathcal{N}_{s_1, \pm\infty}$ are acyclic complexes, we can choose $\tilde{D}_{s_1, -\infty}^{\pm L_2} : A_{s_1, \pm\infty}^-(\vec{L}) \rightarrow A_{s_1, +\infty}^-(\vec{L})$ to be

$$\tilde{D}_{s_1, -\infty}^{\pm L_2} = D(Wh)_{s_1, -\infty}^{\pm L_2} \oplus 0,$$

where $D(Wh)_{s_1, -\infty}^{\pm L_2}$ is the destabilization map for Wh . Therefore $\tilde{\Phi}_s^{\pm L_i} = \Phi(Wh)_s^{\pm L_i} \oplus \Phi_{\mathcal{N}_s}^{\pm L_i}$, where $\Phi_{\mathcal{N}_s}^{\pm L_i} = 0 : \mathcal{N}_s \rightarrow \mathcal{N}_{\psi \pm L_i(s)}$.

Thus the perturbed surgery complex $(\tilde{\mathcal{C}}^-(\vec{L}, \Lambda), \tilde{\mathcal{D}}^-)$ is a direct sum of two twisted gluing of squares

$$(\tilde{\mathcal{C}}^-(\vec{L}, \Lambda), \tilde{\mathcal{D}}^-) = (\mathcal{C}^-(Wh, \Lambda), \mathcal{D}^-) \oplus \prod_{s=(s_1, s_2) \in \mathbb{Z}^2} (\mathcal{N}_s \oplus \mathcal{N}_{s_1, +\infty} \oplus \mathcal{N}_{+\infty, s_2} \oplus \mathcal{N}_{+\infty, +\infty}, \tilde{\mathcal{D}}^-).$$

From the fact that any \mathcal{N}_s with $s \neq (0, 0)$ is acyclic, it follows that $H_*(\tilde{\mathcal{C}}^-(\vec{L}), \tilde{\mathcal{D}}^-) = H_*(\mathcal{C}^-(Wh), \mathcal{D}^-) \oplus H_*(\mathcal{N}_{0,0})$. For that $\mathcal{N}_{0,0}$ belongs to the $(0, 0)$ Spin^c structure and $H_*(\mathcal{N}_{0,0}) = \mathbb{F}[[U]]/U$, we have the equations (6.4). The absolute gradings are inherited from $H_*(\mathcal{C}^-(Wh), \mathcal{D}^-)$. \square

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